

MONGE-AMPÈRE OPERATORS, ENERGY FUNCTIONALS, AND UNIQUENESS OF SASAKI-EXTREMAL METRICS

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ABSTRACT. We develop some pluripotential theoretic techniques for the transversally holomorphic foliation of a Sasakian manifold. We prove the convexity of the K-energy along weak geodesics for Sasakian manifolds. This implies that the K-energy is bounded below if a constant scalar curvature structure exists with those metrics minimizing it. More generally, a relative version of the K-energy is convex, and bounded below if there exists a Sasaki-extremal metric, providing an important necessary condition for Sasaki-extremal metrics. Another application is a proof of the uniqueness of Sasaki-extremal metrics for a fixed transversally holomorphic structure on the Reeb foliation.

1. INTRODUCTION

There has been a renewed interest in Sasakian geometry recently from two sources. First, they have provided a very good source of new examples of Einstein manifolds [12, 10, 33] and the survey article [43]. Second, they play a crucial role in the AdS/CFT correspondence [1, 35, 37, 36], which is a proposed duality between string theory on an odd dimensional Einstein manifold and conformal field theory. It is also worth mentioning that metric cones over Sasakian manifolds arise as the tangent cones at infinity of non-compact Calabi-Yau manifolds with Euclidean volume growth [26, 22].

These new results have been facilitated by the fact that a Sasakian manifold is an odd dimensional contact analogue of a Kähler manifold, both the metric cone over the manifold and the transversal space to the Reeb foliation have natural Kähler structures, so many of the techniques used in Kähler geometry are applicable. In particular, one expects that much of the results in Kähler geometry related to the program proposed by S. Donaldson [24, 25], which was conjectured earlier by S.-T. Yau [48], will hold for Sasakian manifolds, in which the existence and uniqueness of constant scalar curvature Kähler metrics is considered as a problem in infinite dimensional geometric invariant theory. Much of the work was done earlier and independently by T. Mabuchi, and S. Semmes [34, 42], in which the space of Kähler metrics \mathcal{H} in a given Kähler class was shown to have a natural weak Riemannian structure and Riemannian connection. The role of the Kempf-Ness functional in finite dimensional geometric invariant theory is played by the K-energy on \mathcal{H} . It was observed by S. Donaldson that the existence of geodesics in \mathcal{H} would lead to a proof of uniqueness of constant scalar curvature metrics in \mathcal{H} and some sort of convexity of the K-energy should provide necessary and sufficient for existence. Unfortunately, smooth geodesics are not known to exist in \mathcal{H} . But X. X. Chen

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proved the existence of weak $C^{1,1}$ geodesics [20]. This was sufficient for X. X. Chen to prove the uniqueness of constant scalar curvature Kähler metrics when $c_1(M) \leq 0$. Uniqueness was proved in general by X. X. Chen and G. Tian [17] by proving stronger partial regularity on the geodesics. Recently, R. Berman and B. Berndtsson [41] and X. X. Chen, L. Li, and M. Paun [46] proved the geodesic convexity of the K-energy on weak geodesics, giving a simpler proof of uniqueness of constant scalar curvature Kähler metrics.

In Sasakian geometry the Reeb vector field is the analogue of a polarization in Kähler geometry. And \mathcal{H} is the space of transversal Kähler metrics with the given polarization and transversal complex structure. P. Guan and X. Zhang [29] proved the existence of weak $C^{1,1}$ geodesics between elements of \mathcal{H} . They also proved the uniqueness of constant scalar curvature Sasaki(cscS) metrics when $c_1^b(M) \leq 0$, where $c_1^b(M)$ denotes the basic first Chern class of the Reeb foliation. In the Sasaki-Einstein case $c_1^b(M) = a[\omega^T]$, $a > 0$, Y. Nitta and K. Sekiya proved the uniqueness of Sasaki-Einstein metrics, up to automorphisms of the transversal holomorphic structure, by extending the arguments of S. Bando and T. Mabuchi [38]. Uniqueness for toric cscS structures is also known due to K. Cho, A. Futaki, K. Ono [21].

In this article we prove the uniqueness of cscS metrics in \mathcal{H} in general, and more generally, we prove the uniqueness of Sasaki-extremal metrics up to the automorphisms of the Reeb foliation and its transversal holomorphic structure. Sasaki-extremal metrics were first defined by C. Boyer, K. Galicki, S. Simanca [13], termed *canonical Sasakian* metrics. A Sasaki-extremal metric is a critical point of the *Calabi functional*

$$\begin{aligned} \text{Cal}_{M,\xi} : \mathcal{H} &\rightarrow \mathbb{R} \\ (1) \quad \text{Cal}_{M,\xi}(\phi) &:= \int_M (S_\phi - \overline{S})^2 d\mu_\phi, \end{aligned}$$

where $d\mu_\phi = (\omega^T + dd^c\phi)^m \wedge \eta$. As in the Kähler case, extremal metrics are constant scalar curvature precisely when the transversal Futaki invariant vanishes. Thus it enlarges the cases in which a *canonical* metric exists. There has been much research on Sasaki-extremal metrics recently. See [9, 14, 15, 16] for some recent work.

This article will provide the useful uniqueness result and obstructions involving the K-energy. Thus, when they exist Sasaki-extremal metrics provide a canonical Sasakian metric for a given transversely holomorphic foliation. But from work in the Kähler case, we know that such metrics will not always exist.

The central result is convexity of the K-energy along weak $C^{1,1}$ geodesics, denoted $C_w^{1,1}$. See the definition before Theorem 2.4.1. As in [19] we can extend the K-energy $\mathcal{M} : \mathcal{H} \rightarrow \mathbb{R}$ to $\mathcal{H}_{1,1}$, where $\mathcal{H}_{1,1}$ is the space of transversal Kähler potentials $\phi \in C_w^{1,1}$, weak $C^{1,1}$, with $\omega^T + dd^c\phi \geq 0$.

Let $\phi_0, \phi_1 \in \mathcal{H}$, where we consider \mathcal{H} to be the space of smooth transversal Kähler potentials. Let $\phi_t, 0 \leq t \leq 1$, be a weak $C^{1,1}$ geodesic, that is $\phi \in C_w^{1,1}(M \times [0, 1])$ and $\omega^T + dd^c\phi_t \geq 0$ for each $t \in [0, 1]$.

Theorem 1. *The K-energy \mathcal{M} is convex along weak $C_w^{1,1}$ geodesics, that is, $\mathcal{M}(\phi_t)$ is convex in $t \in [0, 1]$.*

The proof of Theorem 1 involves pluripotential theoretic arguments on the transversal space to the Reeb foliation. Much of § 2 is spent developing the necessary background on transversal plurisubharmonic functions, currents, and Monge-Ampère operators on the transversal space. Much of this work is of independent

interest, such as weak continuity of the transversal Monge-Ampère operator and a strong maximal principle. The latter give uniqueness for weak geodesics that are only assumed to be continuous. These results hopefully will provide a useful framework for future work in Sasakian geometry along the lines of the analytical approaches to Kähler geometry such as [4, 6].

In § 3 we define the energy functionals on the space of potentials that will be needed to define the K-energy \mathcal{M} on weak potentials and in proving Theorem 1. Theorem 1 is proved in § 3.3. The main part of the proof is proving that $\mathcal{M}(u_\tau)$ is weakly subharmonic in $\tau \in D \subset \mathbb{C}$ when $\{u_\tau\}$ is a weak geodesic in the domain D .

An important application of Theorem 1 is the proof of uniqueness of constant scalar curvature Sasakian (cscS) structures modulo diffeomorphisms preserving the transversely holomorphic foliation. We denote by $\mathcal{S}(\xi, J)$ the space of Sasakian structures with Reeb vector field ξ and transversely holomorphic structure J . Let $\text{Fol}(\mathcal{F}_\xi, J)$ be the group of diffeomorphisms preserving the Reeb foliation \mathcal{F}_ξ along with its transversely holomorphic structure.

Corollary 2. *Suppose that $(\eta_0, \xi, \omega_0^T), (\eta_1, \xi, \omega_1^T) \in \mathcal{S}(\xi, J)$ are two cscS structures. Then there is a $g \in \text{Fol}(\mathcal{F}_\xi, J)$ so that $g^*\omega_1^T = \omega_0^T$.*

Using basic properties of convex functions we easily prove the following sub-slope inequality.

Corollary 3. *Suppose $\phi_0, \phi_1 \in \mathcal{H}$, then the following inequality holds*

$$\mathcal{M}(\phi_1) - \mathcal{M}(\phi_0) \geq -d(\phi_0, \phi_1)(\text{Cal}_{M,\xi}(\phi_0))^{\frac{1}{2}},$$

where d is the distance function of the Mabuchi metric on \mathcal{H} .

Thus any metric with constant scalar curvature minimizes the K-energy. Furthermore, by Corollary 2 in this case the K-energy achieves its minimum precisely on the orbit of $\text{Fol}(\mathcal{F}_\xi, J)$.

More generally we consider Sasaki-extremal structures. When considering Sasaki-extremal structures it is useful to consider a modified or relative version of the K-energy \mathcal{M}^V . Let $G \subset \text{Fol}(\mathcal{F}_\xi, J)$ be a maximal compact connected subgroup, and (g, η, ξ, Φ) be G -invariant. Then \mathcal{M}^V is restricted to the space \mathcal{H}^G of G -invariant potentials and has critical point precisely the potential corresponding to Sasaki-extremal structures. Here V denotes the extremal vector field which is a transversely holomorphic vector field which depends only on the choice of maximal compact group.

Using the convexity of \mathcal{M}^V along weak geodesics we are able to prove the uniqueness of Sasaki-extremal structures modulo $\text{Fol}(\mathcal{F}_\xi, J)$.

Corollary 4. *Suppose that $(\eta_0, \xi, \omega_0^T), (\eta_1, \xi, \omega_1^T) \in \mathcal{S}(\xi, J)$ are two Sasaki-extremal structures. Then there is an $g \in \text{Fol}(\mathcal{F}_\xi, J)$ so that $g^*\omega_1^T = \omega_0^T$.*

We also have sub-slope inequality for the relative K-energy \mathcal{M}^V .

Corollary 5. *Suppose $\phi_0, \phi_1 \in \mathcal{H}^G$, then the following inequality holds*

$$\mathcal{M}^V(\phi_1) - \mathcal{M}^V(\phi_0) \geq -d(\phi_0, \phi_1)(\text{Cal}_{M,\xi}^G(\phi_0))^{\frac{1}{2}},$$

where d is the distance function of the Mabuchi metric on \mathcal{H} .

Here we use a relative version of the Calabi functional $\text{Cal}_{M,\xi}^G(\phi) = \int_M (S_\phi^G)^2 d\mu_\phi$, where S_ϕ^G is the reduced scalar curvature, which is zero precisely when $\phi \in \mathcal{H}^G$ gives

an extremal structure. Thus a Sasaki-extremal structure is a minimum of the relative K-energy \mathcal{M}^V . Corollary 3 and Corollary 5 provide interesting obstructions to the existence of cscS and Sasaki-extremal structures. The existence of a cscS (respectively Sasaki-extremal) structure requires that the K-energy (respectively relative K-energy) is bounded below and that it achieves its minimum.

Both uniqueness results Corollary 2 and Corollary 4 are a consequence of the convexity of \mathcal{M} and \mathcal{M}^V along weak geodesics. But since these functionals are not known to be strictly convex an additional deformation technique is needed to prove these results. This is done in § 4. This involves deforming \mathcal{M} by adding a strictly convex functional \mathcal{F}^μ

$$\mathcal{M}^{t\mu} = \mathcal{M} + t\mathcal{F}^\mu$$

for small t . Using an implicit function theorem argument we prove that $\mathcal{M}^{t\mu}$ has a path of critical points $\phi_t \in \mathcal{H}$ for $t \in [0, \epsilon]$ for a particular potential ϕ_0 in the orbit of a cscS metric. This is proved in Proposition 4.3.1. A bifurcation technique, due to X. Chen, M. Păun, and Y. Zeng [47], must be used since the differential of the map used has a kernel. Uniqueness then follows from the strict convexity of $\mathcal{M}^{t\mu}$.

The corresponding deformation result for the Sasaki-extremal case is given in Proposition 4.4.3. The proof is similar, but more technicalities involving automorphism groups and the relative K-energy \mathcal{M}^V need to be addressed. Uniqueness again follows from the strict convexity of a deformed functional

$$\mathcal{M}^{V,t\mu} = \mathcal{M}^V + t\mathcal{F}^\mu$$

where here the functional is defined on potentials \mathcal{H}^G .

Uniqueness can be slightly generalized. We say that a manifold with a transversely holomorphic foliation with one dimensional leaves (M, \mathcal{F}, J) is of Sasakian type if it admits a Sasakian structure with (\mathcal{F}, J) , with its transversely holomorphic structure J as its Reeb foliation. The next result shows that for such a foliated manifold (M, \mathcal{F}, J) a compatible Sasaki-extremal structure is unique up to homotheties and varying the contact form by harmonic representatives of $H_b^1(M, \mathbb{R}) = H^1(M, \mathbb{R})$.

Corollary 6. *Suppose $(\eta_0, \xi_0, \omega_0^T), (\eta_1, \xi_1, \omega_1^T)$ are two Sasaki-extremal structures compatible with (M, \mathcal{F}, J) , then there is a $g \in \text{Fol}(M, \mathcal{F}, J)$ and an $a > 0$ so that $g^*\omega_1^T = \omega_0^T$ and $g_*\xi_0 = a^{-1}\xi_1$.*

More precisely, in the Corollary we have

$$g^*(a\eta_1, a^{-1}\xi_1, a\omega_1^T) = (\hat{\eta}_0, \xi_0, \omega_0^T),$$

where $\hat{\eta}_0 = \eta_0 + \alpha$ and α is a harmonic representative of $H_b^1(M, \mathbb{R})$. This result solves the uniqueness problem of Sasaki-extremal structures for a fixed transversely holomorphic foliation, since varying the contact form η with a harmonic representative of an element of $H_b^1(M, \mathbb{R}) = H^1(M, \mathbb{R})$, does not effect the scalar curvature or the transversal metric.

The above techniques give results in the α -twisted setting. This approach has shown promise in tackling problems in Kähler geometry [18], so it is of interest in Sasakian geometry. Let α be a closed, basic, positive $(1, 1)$ -form. The α -twisted transversal scalar curvature of (η, ξ, Φ, g) is

$$(2) \quad S_g^T - \text{tr}_{\omega^T} \alpha.$$

A Sasakian metric is twisted cscS if (2) is a constant, and a Sasakian metric is twisted Sasaki-extremal if (2) is the potential of a transversely holomorphic vector field.

Theorem 7. *Suppose that $(\eta_0, \xi, \omega_0^T), (\eta_1, \xi, \omega_1^T) \in \mathcal{S}(\xi, J)$ are two twisted constant scalar curvature structures. Then $\omega_0^T = \omega_1^T$.*

We are able to prove a partial uniqueness result for twisted Sasaki-extremal structures.

It has been pointed out to the author that some of the same results are in the article of Xishen Jin and Xi Zhang [31], though this work was done entirely independently.

2. SAKIAN GEOMETRY AND TRANSVERSAL SPACE

2.1. Sasakian manifolds. We review some of the properties of Sasakian manifolds that we will use. See the monograph [11] for details.

Definition 2.1.1. *A Riemannian manifold (M, g) is a Sasaki manifold, or has a compatible Sasaki structure, if the metric cone $(C(M), \bar{g}) = (\mathbb{R}_{>0} \times M, dr^2 + r^2 g)$ is Kähler with respect to some complex structure I , where r is the usual coordinate on $\mathbb{R}_{>0}$.*

Thus $\dim M$ is odd and denoted $n = 2m + 1$, while $C(M)$ is a complex manifold with $\dim_{\mathbb{C}} C(M) = m + 1$.

We will identify M with the $\{1\} \times M \subset C(M)$. Let $r\partial_r$ be the Euler vector field on $C(M)$. Using the warped product formulae for the cone metric \bar{g} [40] it is easy check that $r\partial_r$ is real holomorphic, ξ is Killing with respect to both g and \bar{g} , and furthermore the orbits of ξ are geodesics on (M, g) . Define $\eta = \frac{1}{r^2}\xi \lrcorner \bar{g}$, then we have

$$(3) \quad \eta = -\frac{I^* dr}{r} = d^c \log r,$$

where $d^c = \sqrt{-1}(\bar{\partial} - \partial)$. If ω is the Kähler form of \bar{g} , then

$$(4) \quad \omega = \frac{1}{2}d(r^2\eta) = \frac{1}{4}dd^c(r^2).$$

From (4) we have

$$(5) \quad \omega = r dr \wedge \eta + \frac{1}{2}r^2 d\eta.$$

Then (5) implies that η is a contact form with Reeb vector field ξ , since $\eta(\xi) = 1$ and $\mathcal{L}_\xi \eta = 0$. Let $D \subset TM$ be the contact distribution which is defined by

$$(6) \quad D_x = \ker \eta_x$$

for $x \in M$. Furthermore, if we restrict the almost complex structure to D , $J := I|_D$, then (D, J) is a strictly pseudoconvex CR structure on M . We have a splitting of the tangent bundle TM

$$(7) \quad TM = D \oplus L_\xi,$$

where L_ξ is the trivial subbundle generated by ξ . It will be convenient to define a tensor $\Phi \in \text{End}(TM)$ by $\Phi|_D = J$ and $\Phi(\xi) = 0$. Then

$$(8) \quad \Phi^2 = -\mathbb{1} + \eta \otimes \xi.$$

Since ξ is Killing, we have

$$(9) \quad d\eta(X, Y) = 2g(\Phi(X), Y), \quad \text{where } X, Y \in \Gamma(TM),$$

and $\Phi(X) = \nabla_X \xi$, where ∇ is the Levi-Civita connection of g . Making use of (8) we see that

$$g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

and one can express the metric by

$$(10) \quad g(X, Y) = \frac{1}{2}(d\eta)(X, \Phi Y) + \eta(X)\eta(Y).$$

We will denote a Sasaki structure on M by (g, η, ξ, Φ) .

2.2. Transversal holomorphic structure. We now describe a transverse Kähler structure on \mathcal{F}_ξ . The vector field $\xi - \sqrt{-1}I\xi = \xi + \sqrt{-1}r\partial_r$ is holomorphic on $C(M)$. If we denote by $\tilde{\mathbb{C}}^*$ the universal cover of \mathbb{C}^* , then $\xi + \sqrt{-1}r\partial_r$ induces a holomorphic action of $\tilde{\mathbb{C}}^*$ on $C(M)$. The orbits of $\tilde{\mathbb{C}}^*$ intersect $M \subset C(M)$ in the orbits of the Reeb foliation generated by ξ . We denote the Reeb foliation by \mathcal{F}_ξ . This gives \mathcal{F}_ξ a transversely holomorphic structure.

The foliation \mathcal{F}_ξ together with its transverse holomorphic structure is given by an open covering $\{U_\alpha\}_{\alpha \in A}$ of M by product neighborhoods. That is, there are charts

$$(11) \quad \Psi_\alpha : U_\alpha \rightarrow W_\alpha \times (-\epsilon, \epsilon),$$

with $W_\alpha \subset \mathbb{C}^m$, where $\Psi_\alpha(x) = (\phi_\alpha(x), \tau_\alpha)$ and the leaves are locally given by $\phi_\alpha^{-1}(z)$ for $z \in W_\alpha$. And we may assume that ξ is mapped to ∂_t in the coordinates (Z_α, t_α) on $W_\alpha \times (-\epsilon, \epsilon)$. When $U_\alpha \cap U_\beta \neq \emptyset$ the transition maps

$$(12) \quad \Psi_\beta \circ \Psi_\alpha^{-1} : \Psi_\alpha(U_\alpha \cap U_\beta) \rightarrow \Psi_\beta(U_\alpha \cap U_\beta)$$

are given by $\Psi_\beta \circ \Psi_\alpha^{-1}(z, t) = (\phi_\beta \circ \phi_\alpha^{-1}(z), t + \theta_{\beta\alpha}(z))$. Since \mathcal{F}_ξ is *transversely holomorphic* the transitions

$$(13) \quad \phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : W_\alpha \cap W_\beta \rightarrow W_\alpha \cap W_\beta$$

are biholomorphisms, and satisfy the cocycle condition $\phi_{\gamma\beta} \circ \phi_{\beta\alpha} = \phi_{\gamma\alpha}$ on $W_\alpha \cap W_\beta \cap W_\gamma$. The transversal Kähler form ω^T induces a Kähler form ω_α on W_α with $\phi_{\beta\alpha}^* \omega_\beta = \omega_\alpha$.

In working on the transversal space we work on the charts $\{W_\alpha\}_{\alpha \in A}$ with their Kähler structure invariant under the transitions $\phi_{\beta\alpha}$. But it will also be useful to consider *basic* functions and tensors. If we define $\nu(\mathcal{F}_\xi) = TM/L_\xi$ to be the normal bundle to the leaves, then we can generalize the above concept.

Definition 2.2.1. A tensor $\Psi \in \Gamma((\nu(\mathcal{F}_\xi)^*)^{\otimes p} \otimes \nu(\mathcal{F}_\xi)^{\otimes q})$ is basic if $\mathcal{L}_V \Psi = 0$ for any vector field $V \in \Gamma(L_\xi)$.

It is sufficient to check this for $V = \xi$. Then g^T and ω^T are such tensors on $\nu(\mathcal{F}_\xi)$. We will also make use of the bundle isomorphism $\pi : D \rightarrow \nu(\mathcal{F}_\xi)$, which induces an almost complex structure \bar{J} on $\nu(\mathcal{F}_\xi)$ so that $(D, J) \cong (\nu(\mathcal{F}_\xi), \bar{J})$ as complex vector bundles. Clearly, \bar{J} is basic and is mapped by the foliation charts ϕ_α to the complex structure on W_α . In the sequel we will denote the almost complex structure on $\nu(\mathcal{F}_\xi)$ by J and denote the Reeb foliation with its transversal holomorphic structure by (\mathcal{F}_ξ, J) .

Smooth basic functions will be denoted by $C_b^\infty(M)$. The basic exterior r-forms are denoted Ω_b^r , and split into types

$$\Omega_b^r = \bigoplus_{p+q=r} \Omega_b^{p,q}.$$

To work on the Kähler leaf space we define the Levi-Civita connection of g^T by

$$(14) \quad \nabla_X^T Y = \begin{cases} \pi_\xi(\nabla_X Y) & \text{if } X, Y \text{ are smooth sections of } D, \\ \pi_\xi([V, Y]) & \text{if } X = V \text{ is a smooth section of } L_\xi, \end{cases}$$

where $\pi_\xi : TM \rightarrow D$ is the orthogonal projection onto D . Then ∇^T is the unique torsion free connection on $D \cong \nu(\mathcal{F}_\xi)$ so that $\nabla^T g^T = 0$. Then for $X, Y \in \Gamma(TM)$ and $Z \in \Gamma(D)$ we have the curvature of the transverse Kähler structure

$$(15) \quad R^T(X, Y)Z = \nabla_X^T \nabla_Y^T Z - \nabla_Y^T \nabla_X^T Z - \nabla_{[X, Y]}^T Z,$$

and similarly we have the transverse Ricci curvature Ric^T and scalar curvature S^T .

The following follows from O'Neill tensor computations for a Riemannian submersion. See [39] and [7, Ch. 9].

Proposition 2.2.2. *Let (M, g, η, ξ, Φ) be a Sasaki manifold of dimension $n = 2m + 1$, then*

- (i) $\text{Ric}_g(X, \xi) = 2m\eta(X)$, for $X \in \Gamma(TM)$,
- (ii) $\text{Ric}^T(X, Y) = \text{Ric}_g(X, Y) + 2g^T(X, Y)$, for $X, Y \in \Gamma(D)$,
- (iii) $S^T = S_g + m$.

Here we define S_g (respectively S^T) to be $1/2$ the trace of Ric_g with respect to g (respectively $1/2$ the trace of Ric^T with respect to g^T) to simplify notation later on.

We define $\mathcal{S}(\xi, J)$ to be the set of Sasakian structures with Reeb vector field ξ and with the holomorphic structure J on the Reeb foliation \mathcal{F}_ξ . In other words, the set of Sasakian structures inducing the same complex normal bundle $(\nu(\mathcal{F}_\xi), J)$. This is the set of $(\tilde{g}, \tilde{\eta}, \xi, \tilde{\Phi}) \in \mathcal{S}(\xi)$ such that the following diagram commutes

$$(16) \quad \begin{array}{ccc} TM & \xrightarrow{\tilde{\Phi}} & TM \\ \downarrow & & \downarrow \\ \nu(\mathcal{F}_\xi) & \xrightarrow{J} & \nu(\mathcal{F}_\xi). \end{array}$$

The next lemma describes $\mathcal{S}(\xi, J)$ in detail. Define

$$\mathcal{H}_{\omega^T} = \{\phi \in C_b^\infty(M) \mid (\omega^T + dd^c \phi)^m \wedge \eta > 0\}$$

Lemma 2.2.3 ([11, 13]). *The space $\mathcal{S}(\xi, J)$ of all Sasaki structures with Reeb vector field ξ and transverse holomorphic structure J is an affine space modeled on $\mathcal{H}/\mathbb{R} \times C_b^\infty(M)/\mathbb{R} \times H^1(M, \mathbb{R})$. If $(g, \eta, \xi, \Phi) \in \mathcal{S}(\xi, J)$ is a fixed Sasaki structure then another structure $(\tilde{g}, \tilde{\eta}, \xi, \tilde{\Phi}) \in \mathcal{S}(\xi, J)$ is determined by real basic functions ϕ and ψ and an harmonic, with respect to g , 1-form α such that*

$$(17) \quad \begin{aligned} \tilde{\eta} &= \eta + 2d^c \phi + d\psi + \alpha, \\ \tilde{\Phi} &= \Phi - \xi \otimes \tilde{\eta} \circ \Phi, \\ \tilde{g} &= \frac{1}{2} d\tilde{\eta} \circ (\mathbb{1} \otimes \tilde{\Phi}) + \tilde{\eta} \otimes \tilde{\eta}, \end{aligned}$$

and the transversal Kähler form becomes $\tilde{\omega}^T = \omega^T + dd^c\phi$.

Proof. We give only a sketch. See [11] for details. The 1-form $\gamma = \tilde{\eta} - \eta$ is basic, and since $d\gamma \in \Gamma(\Lambda_b^{1,1})$ and γ is real, $d^c d\gamma = 0$. And we have the Hodge decomposition

$$(18) \quad \gamma = d^c\phi + d\psi + \alpha,$$

with respect to the transversal Kähler metric g^T , where $\alpha \in \mathcal{H}_{g^T}^1$ is harmonic. But note that $\mathcal{H}_{\mathbb{R}, g^T}^1 = \mathcal{H}_{\mathbb{R}, g}^1$, where the latter is the space of real harmonic 1-forms on (M, g) . This is because a $\beta \in \Gamma(\Lambda^1(M))$ satisfying $d\beta = 0$ and $\mathcal{L}_\xi\beta = 0$ must be basic. \square

It is easy to check that the parameter ψ in (17) changes the structure only by a gauge transformation along the leaves. That is, if $\psi \in C_b^\infty(M)$, then $\exp(\psi\xi)^*\eta = \eta + d\psi$. Altering by a harmonic form likewise does not effect the transversal metric, so it will not be of much interest.

Given $\phi \in \mathcal{H}_{\omega^T}$ we define the *transversal Kähler deformation* of (g, η, ξ, Φ) to be the Sasakian structure given in the lemma, which we will denote $(g_\phi, \eta_\phi, \xi, \Phi_\phi)$. We will denote by $\omega_\phi^T = \omega^T + dd^c\phi$ the transversal Kähler form of the deformed structure.

This article is concerned with constant scalar curvature Sasakian structures (cscS) and more generally Sasaki-extremal structures. By Proposition 2.2.2 the scalar curvature S_g and the scalar curvature of the transversal structure S_g^T differ by a constant, both these conditions are given by the transversal Kähler structure.

Let $\mathfrak{fol}(M, \mathcal{F}_\xi, J)$ be the space of vector fields preserving the Reeb foliation along with its transversal holomorphic structure. The corresponding group is denoted by $\text{Fol}(M, \mathcal{F}_\xi, J)$. This is an infinite dimensional group, since any vector field tangent to the leaves is in $\mathfrak{fol}(M, \mathcal{F}_\xi, J)$. So we define $\mathfrak{hol}^T(\xi, J)$ to be the image of

$$(19) \quad \begin{array}{ccc} \mathfrak{fol}(M, \mathcal{F}_\xi, J) & \xrightarrow{\pi} & \Gamma(\nu(\mathcal{F}_\xi)) \\ X & \mapsto & \overline{X} \end{array}$$

which is a finite dimensional complex Lie algebra. We will use $\mathfrak{hol}^T(\xi, J)$ to denote both transversally holomorphic $(1, 0)$ vector fields, or transversally real holomorphic vector fields depending on the context.

Given a basic $\phi \in C_b^\infty(M, \mathbb{C})$, we define $\partial_g^\# \phi$ to be the $(1, 0)$ component of the gradient, that is

$$(20) \quad g(\partial_g^\# \phi, \cdot) = \bar{\partial}\phi.$$

In order for $\partial_g^\# \phi \in \mathfrak{hol}^T(\xi, J)$, transversely holomorphic, we need in addition $\bar{\partial}_b \partial_g^\# \phi = 0$. This is equivalent to the fourth-order transversally elliptic equation

$$(21) \quad L_g \phi := (\bar{\partial} \partial_g^\#)^* \bar{\partial} \partial_g^\# \phi.$$

We have

$$(22) \quad L_g \phi = \frac{1}{4} \Delta_b^2 \phi + \frac{1}{2} (\rho^T, dd^c \phi) + (\partial S^T) \lrcorner \partial_g^\# \phi.$$

We define the space of *holomorphy potentials* to be $\mathcal{H}_g := \ker L_g$.

We define the *Calabi functional*

$$\text{Cal}_{M, \xi} : \mathcal{S}(\xi, J) \rightarrow \mathbb{R}$$

$$(23) \quad \text{Cal}_{M,\xi}(g) := \int_M (S_g - \overline{S})^2 d\mu_g,$$

where $d\mu_g = (\omega^T)^m \wedge \eta$.

Definition 2.2.4. A Sasakian structure (g, η, ξ, Φ) is Sasaki-extremal if it is a critical point of the Calabi functional. Equivalently, $\partial^\# S_g$ is transversally holomorphic.

See [13] for the proof that the two conditions are equivalent.

A function $u : W \rightarrow \mathbb{R} \cup \{-\infty\}$ on an open set $W \subset \mathbb{C}^m$ is *quasi-plurisubharmonic* if it can locally be written as the sum of a plurisubharmonic function and a smooth function. Recall that a function u on W is plurisubharmonic if

- (i) u is upper semicontinuous;
- (ii) for every complex line $L \subset \mathbb{C}^m$, $u|_{L \cap W}$ is subharmonic on $L \cap W$.

Let θ be any closed basic $(1, 1)$ -form.

Definition 2.2.5. A function $u : M \rightarrow \mathbb{R} \cup \{-\infty\}$ is said to be transversally θ -plurisubharmonic (θ -psh) if u is invariant under the Reeb flow, is upper semicontinuous, in each foliation chart W_α u is quasi-plurisubharmonic and

$$\theta + dd^c u \geq 0,$$

as a $(1, 1)$ -current.

The set of θ -psh functions on M is denoted $\text{PSH}(M, \theta)$.

Because a function $u \in \text{PSH}(M, \theta)$ is defined to be θ -psh in each holomorphic foliation chart W_α most of the familiar properties translate into this situation. For example $u \in L^1(d\mu_\eta)$ where $d\mu_\eta = (\omega^T)^m \wedge \eta$. See [23].

The following approximation result will be useful.

Proposition 2.2.6. Suppose θ is a positive basic $(1, 1)$ -form on M . Let $\phi \in \text{PSH}(M, \theta) \cap C^0(M)$, then there exists a sequence $\phi_j \in \text{PSH}(M, \theta)$ decreasing to ϕ .

The proof will mostly follow from the following.

Theorem 2.2.7 ([8]). Let X be a complex manifold with a positive hermitian form ω and γ a continuous $(1, 1)$ -form on M . Let $\phi \in \text{PSH}(X, \gamma)$ be locally bounded. Then for any relatively compact open $X' \subset X$ we have a decreasing sequence $\varepsilon_j \searrow 0$ and a sequence $\phi_j \in \text{PSH}(X', \gamma + \varepsilon_j \omega) \cap C^\infty(X')$ decreasing to ϕ .

Proof of Proposition. Let $X = C(M)$ with $M = \{r = 1\} \subset X$. And let $X' \subset X$ be $X' = \{(r, x) \in C(M) \mid 1 - \epsilon < r < 1 + \epsilon\}$.

For $\phi \in \text{PSH}(M, \theta) \cap C^0(M)$ consider ϕ to be a function on X via the projection $p : X \rightarrow M$, $p(r, x) = x$, and similarly consider θ as a $(1, 1)$ -form on X . Define

$$\omega = \frac{dr}{r} \wedge \eta + \theta$$

a positive hermitian form on X . By the theorem there exists $\psi_j \in \text{PSH}(X', \theta + \varepsilon_j \omega) \cap C^\infty(X')$ with $\psi_j \searrow \phi$. Let $T \subset \text{Aut}(M, \eta, \xi, g)$ be the torus generated by ξ . By averaging by T we may assume that ψ_j are invariant by ξ .

Suppose that $\psi \in C^\infty(X')$ is invariant under ξ . Routine calculation shows that the complex hessian at a point of $M \subset X'$ for $X, Y \in \Gamma(D)$ basic is

$$(24) \quad dd^c \psi(X, Y) = d_b d_b^c \psi(X, Y) + 2d\psi(\partial_r) \omega^T(X, Y).$$

On X' we have

$$(25) \quad \varepsilon_j \frac{dr}{r} \wedge \eta + (1 + \varepsilon_j)\theta + dd^c \psi_j \geq 0.$$

Introduce the coordinate $t = \log r$ so $\partial_t = r\partial_r$. Then substituting (∂_t, ξ) into the above inequality gives $\varepsilon_j + \partial_t^2 \psi_j \geq 0$. Then $\hat{\psi}_j = \psi_j + \varepsilon_j t^2$ is convex with respect to t and converges uniformly to ϕ on $[-\delta, \delta] \times M = \{(t, x) \in X' \mid -\delta < t < \delta\} \subset X'$. By convexity we have

$$\frac{\hat{\psi}_j(0, x) - \hat{\psi}_j(-\delta, x)}{\delta} \leq \partial_t \psi_j(0, x) \leq \frac{\hat{\psi}_j(\delta, x) - \hat{\psi}_j(0, x)}{\delta}$$

Or

$$\frac{\psi_j(0, x) - \psi_j(-\delta, x) - \varepsilon_j \delta^2}{\delta} \leq \partial_t \psi_j(0, x) \leq \frac{\psi_j(\delta, x) - \psi_j(0, x) + \varepsilon_j \delta^2}{\delta}.$$

Thus $\partial_t \psi_j \rightarrow 0$ uniformly on $M \subset X'$.

From (24) and (25) on M we have

$$(1 + \varepsilon_j)\theta + d_b d_b^c \psi_j + 2d\psi_j(\partial_t)\omega^T \geq 0.$$

After possibly passing to a subsequence of ψ_j there are constants $\hat{\varepsilon}_j \searrow 0$ with

$$(1 + \hat{\varepsilon}_j)\theta + d_b d_b^c \psi_j \geq 0.$$

Without loss of generality we may assume that $\phi \leq -1$. Then we have $\lambda_j = 1 + \hat{\varepsilon}_j$, $\lambda_j \searrow 1$, and negative $\psi_j \in \text{PSH}(M, \lambda_j \theta)$. Then $\phi_j := \psi_j / \lambda_j \in \text{PSH}(M, \theta) \cap C^\infty(M)$ is a sequence decreasing to ϕ . \square

2.3. Monge-Ampère operator. We will define a transversal version of the Monge-Ampère operator on transversally quasi-plurisubharmonic functions on M . This will be needed to define weak geodesics in the space of Sasakian structures. It will also be needed to define necessary energy functionals on weak structures, in particular the Monge-Ampère energy and the Mabuchi K-energy.

We can define a *transversal current* T on the foliation \mathcal{F}_ξ to be a collection $\{W_\alpha, T_\alpha\}_{\alpha \in A}$ so that $\phi_{\beta\alpha}^* T_\alpha|_{W_\alpha \cap W_\beta} = T_\beta|_{W_\alpha \cap W_\beta}$. Since that transition maps (13) are holomorphic, we define T to have bidegree (p, q) if each T_α has bidegree (p, q) , $T_\alpha \in \mathcal{D}^{p,q}(W_\alpha)$. Similarly, we define the notions of a closed transversal current, respectively a positive current, to be transversal currents with each (W_α, T_α) closed, respectively positive.

If θ is any basic closed $(1, 1)$ -form, and $u \in \text{PSH}(M, \theta)$, then $\theta + dd^c u$ is a closed positive $(1, 1)$ transversal current. Suppose that $u \in \text{PSH}(M, \theta) \cap L^\infty$ and T is a closed positive transversal current of bidegree (p, p) . One can employ the Bedford-Taylor [3] construction to define the closed, positive, degree $(p+1, p+1)$ transversal current

$$(\theta + dd^c u) \wedge T.$$

This is of course defined in each chart W_α , as follows. If $\theta = dd^c w$, then $dd^c(w + u) \wedge T_\alpha := dd^c((w + u)T_\alpha)$. One can check that this is independent of w .

Given $u_1, \dots, u_m \in \text{PSH}(M, \theta) \cap L^\infty$ by applying this definition inductively, we get

$$(\theta + dd^c u_1) \wedge \dots \wedge (\theta + dd^c u_m),$$

a positive, bidegree (m, m) current defined in each W_α . It therefore defines a Radon measure in each W_α , invariant under the transitions (13).

We define the *Monge-Ampère operator* as follows. We define the measure, denoted

$$(26) \quad (\theta + dd^c u_1) \wedge \cdots \wedge (\theta + dd^c u_m) \wedge \eta,$$

to be the product measure on $U_\alpha \cong W_\alpha \times (-\epsilon, \epsilon)$ given by the chart (11). It is easy to see that this is invariant of the transition maps (12). This is easy to see using Fubini's theorem, and the invariance of $(\theta + dd^c u_1) \wedge \cdots \wedge (\theta + dd^c u_m)$ under the holomorphic transitions (13).

The Monge-Ampère operator is defined as locally a product, so many of the usual properties of the usual Monge-Ampère operator on complex manifolds hold. The most important will be weak convergence under several cases of convergence of function.

Theorem 2.3.1. *Let u_0^j be a sequence of bounded transversally quasi-psh functions, and sequences $u_1^j, \dots, u_m^j \in \text{PSH}(M, \theta) \cap L^\infty$. Then*

$$u_0^j(\theta + dd^c u_1^j) \wedge \cdots \wedge (\theta + dd^c u_m^j) \wedge \eta$$

converges weakly to

$$u_0(\theta + dd^c u_1) \wedge \cdots \wedge (\theta + dd^c u_m) \wedge \eta,$$

where u_0 is transversal quasi-psh and bounded and $u_1, \dots, u_m \in \text{PSH}(M, \theta) \cap L^\infty$, when the convergence $u_k^j \rightarrow u_k$ for each k is one of the following.

- u_k^j decreases pointwise to u_k .
- u_k^j increases to u_k a.e. with respect to Lebesgue measure.
- u_k^j converges to u_k uniformly on M .

We note that if T is a closed positive transversal current of bidegree $(m-1, m-1)$ and u, v are bounded transversal quasi-psh then $du \wedge d^c v \wedge T \wedge \eta$ can be defined. We may suppose $u \geq 0$ and define

$$du \wedge d^c u \wedge T \wedge \eta := \frac{1}{2} dd^c u^c \wedge T - u dd^c u \wedge T \wedge \eta,$$

and the general case can be defined by polarization. In particular, $du \wedge d^c u \wedge T \wedge \eta \geq 0$ and we have the analogous convergence as in Theorem 2.3.1.

Integration by parts formulae will be useful.

Proposition 2.3.2. *Suppose that θ is a positive, closed, basic $(1, 1)$ -form on M . Let v, w each be differences of continuous transversally quasi-psh functions, and let $u_1, \dots, u_{m-1} \in \text{PSH}(M, \theta) \cap C^0(M)$. Then*

$$(27) \quad \begin{aligned} & \int_M v dd^c w \wedge (\theta + dd^c u_1) \wedge \cdots \wedge (\theta + dd^c u_{m-1}) \wedge \eta \\ &= \int_M w dd^c v \wedge (\theta + dd^c u_1) \wedge \cdots \wedge (\theta + dd^c u_{m-1}) \wedge \eta \\ &= - \int_M dv \wedge d^c w \wedge (\theta + dd^c u_1) \wedge \cdots \wedge (\theta + dd^c u_{m-1}) \wedge \eta. \end{aligned}$$

Proof. By assumption $v = q - r, w = s - t$ with q, r, s, t quasi-psh. By Proposition 2.2.6 $q, r, s, t, u_1, \dots, u_{m-1}$ can be approximated by decreasing sequences of smooth transversally quasi-psh functions. The above equations hold for these approximations by Stoke's theorem. Then the result follows from Theorem 2.3.1. \square

2.4. Weak geodesics. We are primarily concerned with applications of the Monge-Ampère operator to weak geodesics in \mathcal{H} . We describe the weak Riemannian structure on \mathcal{H} . Given $\phi \in \mathcal{H}$ and $\psi_1, \psi_2 \in T_\phi \mathcal{H} \cong C_b^\infty(M)$

$$\langle \psi_1, \psi_2 \rangle_\phi := \int_M \psi_1 \psi_2 d\mu_\phi,$$

where $d\mu_\phi = (\omega_\phi^T)^m \wedge \eta_\phi$.

There is a torsion free connection compatible with this metric. Given a smooth path $\{\phi_t | a \leq t \leq b\}$ in \mathcal{H} a vector field along $\{\phi_t\}$ can be identified with a smooth path $\{\psi_t | a \leq t \leq b\} \in C_b^\infty([a, b] \times M)$. The covariant derivative, in transversal holomorphic coordinates (11), is

$$\frac{D}{\partial t} := \partial_t - \frac{1}{2\sqrt{-1}} \sum (\omega_\phi^T)^{\alpha\bar{\beta}} (\dot{\phi}_\alpha \partial_{\bar{\beta}} + \dot{\phi}_{\bar{\beta}} \partial_\alpha).$$

The geodesic equation is then

$$(28) \quad \ddot{\phi} = \frac{1}{2} |d\dot{\phi}|_{\omega_\phi^T}^2,$$

for a smooth path $\{\phi_t | a \leq t \leq b\} \subset \mathcal{H}$.

We define the *Monge-Ampère energy* to be the potential $\mathcal{E} : \mathcal{H} \rightarrow \mathbb{R}$ with derivative

$$d\mathcal{E}|_\phi(\psi) = \int_M \psi d\mu_\phi,$$

which is easily seen to be closed. So we may define

$$(29) \quad \mathcal{E}(\phi) = \int_0^1 \int_M \dot{\phi}_t d\mu_{\phi_t} dt,$$

where $\{\phi_t | 0 \leq t \leq 1\}$ is a smooth path in \mathcal{H} with $\phi_0 = 0, \phi_1 = \phi$.

Define

$$(30) \quad \tilde{\mathcal{H}} = \{\phi \in \mathcal{H} \mid \mathcal{E}(\phi) = 0\} \subset \mathcal{H}.$$

The map

$$\begin{aligned} \mathcal{H} &\cong \tilde{\mathcal{H}} \times \mathbb{R} \\ \phi &\leftrightarrow \left(\phi - \frac{\mathcal{E}(\phi)}{\text{Vol}(M)}, \frac{\mathcal{E}(\phi)}{\text{Vol}(M)} \right) \end{aligned}$$

is an isometry. And $\tilde{\mathcal{H}}$ is geodesically convex, that is a geodesic $\{\phi_t \mid a \leq t \leq b\}$ with $\phi_a, \phi_b \in \tilde{\mathcal{H}}$ is contained in $\tilde{\mathcal{H}}$.

If \mathcal{K} denotes the space of Sasakian structures associated to \mathcal{H} , then we have an isomorphism

$$\begin{aligned} \tilde{\mathcal{H}} &\cong \tilde{\mathcal{K}} \\ \phi &\leftrightarrow (\eta_\phi, \xi, \Phi_\phi, g_\phi) \end{aligned}$$

A geodesic in \mathcal{K} is defined to be a geodesic in $\tilde{\mathcal{H}}$.

Let $A = \{\tau \in \mathbb{C} \mid 1 \leq |\tau| \leq e\}$, then $N := M \times A$ is a manifold with boundary, with a transversely holomorphic foliation. The foliation charts are as in (11). If $V \subset A$, then the charts are

$$(31) \quad \Phi_\alpha : U_\alpha \times V \rightarrow W_\alpha \times V \times (-\epsilon, \epsilon)$$

with $W_\alpha \times V$ giving the local holomorphic leaf space.

A path $\phi \in C_b^\infty([0, 1] \times M)$ corresponds to an S^1 -invariant function Φ_τ on N under $\tau = e^t$. If $\{\phi_t\}$ is a smooth path in \mathcal{H} then a routine calculation shows that

$$(\pi^*\omega^T + dd^c\Phi)^{m+1} = \frac{(m+1)}{4}(\ddot{\phi}_t - \frac{1}{2}|d\dot{\phi}_t|_{\omega_{\phi_t}^T}^2)(\omega_{\phi_t}^T)^m \wedge \frac{d\tau \wedge d\bar{\tau}}{|\tau|^2}.$$

Thus a smooth geodesic between $\phi_0, \phi_1 \in \mathcal{H}$ is given by $\Phi \in C_b^\infty(N)$ with $\Phi_\tau = \phi_j, |\tau| = e^j, j = 0, 1$, and $\omega^T + dd^c\Phi_\tau > 0$ for all $\tau \in A$ so that

$$(\pi^*\omega^T + dd^c\Phi)^{m+1} = 0.$$

We can define a *weak geodesic* between $\phi_0, \phi_1 \in \mathcal{H}$ as follows.

$$(32) \quad \begin{cases} \Phi \in \text{PSH}(\overset{\circ}{N}, \pi^*\omega^T) \cap C^0(N) \\ \Phi_\tau = \phi_j, |\tau| = e^j, j = 0, 1 \\ (\pi^*\omega^T + dd^c\Phi)^{m+1} \wedge \eta = 0 \end{cases}$$

We assume $\Phi \in C^0(N)$ merely because it is the weakest regularity we will consider.

The best regularity for a solution of the Dirichlet problem (32) is due to P. Guan and Xi Zhang [29]. We define $C_w^{1,1}(N)$ to be the completion of $C_b^\infty(N)$ with norm $\|\phi\|_w = \|\phi\|_{C^1} + \|dd^c\phi\|_{L^\infty}$.

Theorem 2.4.1 ([29]). *The Dirichlet problem (32) for $\phi_0, \phi_1 \in \mathcal{H}$ has a unique solution $\Phi \in C_w^{1,1}(N)$.*

We only have weak regularity to the non-elliptic problem (32), so for $\epsilon > 0$ we define a path $\{\phi_t \mid 0 \leq t \leq 1\}$ in \mathcal{H} to be an ϵ -geodesic if

$$(33) \quad (\ddot{\phi} - \frac{1}{2}|d\dot{\phi}|_{\omega_\phi^T}^2)(\omega_\phi^T)^m = \epsilon(\omega^T)^m.$$

An ϵ -geodesic is necessarily a smooth path in \mathcal{H} , since it is a solution to the transversely elliptic problem

$$(\pi^*\omega^T + dd^c\Phi)^{m+1} = \frac{\epsilon}{4}(\pi^*\omega^T + \frac{\sqrt{-1}d\tau \wedge d\bar{\tau}}{|\tau|^2})^{m+1}.$$

It follows from the proof in [29] and the maximal principle proved below that there are smooth ϵ -geodesics Φ^ϵ monotonically decreasing in $\epsilon > 0$ and $\Phi^\epsilon \rightarrow \Phi$, the weak solution in Theorem 2.4.1, weakly in $C_w^{1,1}(N)$ as $\epsilon \rightarrow 0$.

2.5. Maximal principle and uniqueness results. We will prove a maximal principle for the Monge-Ampère operator on N and some uniqueness results. First we give a version of Proposition 2.3.2 for N .

Proposition 2.5.1. *Let θ be a basic positive $(1,1)$ -form on N . Let v, w each be differences of continuous transversally quasi-psh functions, and let $u_1, \dots, u_m \in \text{PSH}(N, \pi^*\theta) \cap C^0(N)$. Then*

$$(34) \quad \begin{aligned} \int_N v dd^c w \wedge (\pi^*\theta + dd^c u_1) \wedge \dots \wedge (\pi^*\theta + dd^c u_m) \wedge \eta \\ = \int_N w dd^c v \wedge (\pi^*\theta + dd^c u_1) \wedge \dots \wedge (\pi^*\theta + dd^c u_m) \wedge \eta \\ = - \int_N dv \wedge d^c w \wedge (\pi^*\theta + dd^c u_1) \wedge \dots \wedge (\pi^*\theta + dd^c u_m) \wedge \eta, \end{aligned}$$

provided one of v, w , or $T = (\pi^\theta + dd^c u_1) \wedge \dots \wedge (\pi^*\theta + dd^c u_m)$ has compact support in $N \setminus \partial N$.*

Proof. Suppose $v = q - r$ has compact support where q, r are quasi-psh, and we may assume $q, r \leq -1$. Let $f(\tau) = (\log |\tau|)^2 - \log |\tau|$ a strictly psh function on A vanishing on ∂A . Choose $M > 0$ large enough that $q, r > Mf$ outside of the interior of $U = \{x \in N \mid q(x) = r(x)\}$. If we define $\tilde{q} = \max\{q, Mf\}$ and $\tilde{r} = \max\{r, Mf\}$. Let $W \subset N \setminus \partial N$ be a relatively compact open neighborhood containing $\{\tilde{q} \geq Mf\} \cup \{\tilde{r} \geq Mf\}$. Proposition 2.2.6 gives decreasing sequences of smooth transversely quasi-psh q_i and r_i on W with $q_i \searrow q$ and $r_i \searrow r$. The sequences can be chosen so that $q_i, r_i < Mf$ near ∂W . Define $\tilde{q}_i = \max\{q_i, Mf\}$ and $\tilde{r}_i = \max\{r_i, Mf\}$, where we make take the regularized maximum (See [23, I-5.18]) so that \tilde{q}_i, \tilde{r}_i are smooth. Similarly, for each $k = 1, \dots, m$ choose a sequence $u_k^i \in \text{PSH}(W, \pi^*\theta) \cap C_b^\infty(W)$ with $u_k^i \searrow u_k$. And if $w = s - t$ with s, t quasi-psh, we choose sequences s_i, t_i of smooth quasi-psh on W .

The integration by parts formula then holds with $v_i = \tilde{q}_i - \tilde{r}_i, w_i = s_i - t_i$ and u_1^i, \dots, u_m^i substituted by Stoke's theorem, since v_i has compact support in \overline{W} . Applying Theorem 2.3.1 finishes the proof. \square

We prove weak maximal principle first. Let θ be a basic positive $(1, 1)$ -form on N .

Proposition 2.5.2. *Let $u, v \in \text{PSH}(N, \pi^*\theta) \cap C^0(N)$ satisfy $u \leq v$ on ∂N . Then*

$$\int_{v < u} (\pi^*\theta + dd^c u)^{m+1} \wedge \eta \leq \int_{v < u} (\pi^*\theta + dd^c v)^{m+1} \wedge \eta.$$

Proof. Let $\delta > 0$ then $\Omega := \{v < u - \delta\} \Subset N \setminus \partial N$. Define $u^\epsilon := \max\{u - \delta, v + \epsilon\}$ for small $\epsilon > 0$, so $u^\epsilon = v + \epsilon$ in a neighborhood of $\partial\Omega$. We have

$$\int_{\Omega} (\pi^*\theta + dd^c u^\epsilon)^{m+1} \wedge \eta = \int_{\Omega} (\pi^*\theta + dd^c v)^{m+1} \wedge \eta.$$

This is because

$$(35) \quad (\pi^*\theta + dd^c u^\epsilon)^{m+1} \wedge \eta - (\pi^*\theta + dd^c v)^{m+1} \wedge \eta = dd^c(u^\epsilon - v) \wedge T \wedge \eta$$

where $T = \sum_{j=0}^m (\pi^*\theta + dd^c u^\epsilon)^j \wedge (\pi^*\theta + dd^c v)^{m-j}$. Then the integral of (35) is zero by Proposition 2.5.1. Since u^ϵ decreases to $u - \delta$ on Ω , weak convergence of measures gives

$$\int_{\{v < u - \delta\}} (\pi^*\theta + dd^c u)^{m+1} \wedge \eta \leq \int_{\{v < u - \delta\}} (\pi^*\theta + dd^c v)^{m+1} \wedge \eta.$$

The result then follows by taking $\delta \rightarrow 0$ and applying monotone convergence. \square

A consequence is that solutions $u \in \text{PSH}(N, \pi^*\theta) \cap C^0(N)$ to the transversal homogeneous Monge-Ampère equation

$$(36) \quad (\pi^*\theta + dd^c u)^{m+1} \wedge \eta = 0$$

are maximal. By *maximal* we mean that given any neighborhood $U \subset N$ invariant under the Reeb flow, if $v \in \text{PSH}(U, \pi^*\theta) \cap C^0(\overline{U})$ satisfies $v \leq u$ on ∂U , then $v \leq u$ on U . For suppose v satisfies $v \leq u$ on ∂U , then

$$\tilde{u} = \begin{cases} u & \text{on } N \setminus U \\ \max\{u, v\} & \text{on } U \end{cases} \in \text{PSH}(N, \pi^*\theta) \cap C^0(N).$$

We may replace $\pi^*\theta$ with $\tilde{\theta} = \pi^*\theta + dd^c f = \pi^*\theta + \frac{\sqrt{-1}d\tau \wedge d\bar{\tau}}{|\tau|^2}$ and subtract f from u and v . Thus we may assume that $u \in \text{PSH}(N, \tilde{\theta}) \cap C^0(N)$ with $\tilde{\theta}$ a basic strictly positive, closed, $(1, 1)$ -form. For $\delta > 0$ small

$$\begin{aligned} \int_{\{u < (1-\delta)\tilde{u}\}} (\delta\tilde{\theta})^{m+1} \wedge \eta &\leq \int_{\{u < (1-\delta)\tilde{u}\}} (\tilde{\theta} + (1-\delta)dd^c\tilde{u})^{m+1} \wedge \eta \\ &\leq \int_{\{u < (1-\delta)\tilde{u}\}} (\tilde{\theta} + dd^c u)^{m+1} \wedge \eta = 0. \end{aligned}$$

This clearly implies uniqueness of continuous solutions $u \in \text{PSH}(N, \pi^*\theta) \cap C^0(N)$ to (36) with fixed $u|_{\partial N} \in C^0(\partial N)$. This also follows from the *strong maximal principle* which we prove next.

Theorem 2.5.3. *Let $u, v \in \text{PSH}(N, \pi^*\theta) \cap C^0(N)$. Suppose that*

$$(\pi^*\theta + dd^c v)^{m+1} \wedge \eta \leq (\pi^*\theta + dd^c u)^{m+1} \wedge \eta$$

and $u \leq v$ on ∂N . Then $u \leq v$ on N .

Proof. Let $u^\epsilon := \max u, v + \epsilon$, then $u^\epsilon = v + \epsilon$ near ∂N . The following formula, due to J-P Demailly: for $s, w \in \text{PSH} \cap L_{loc}^\infty$

$$(dd^c \max\{s, w\})^{m+1} \geq \mathbb{1}_{\{s \geq w\}} (dd^c s)^{m+1} + \mathbb{1}_{\{s < w\}} (dd^c w)^{m+1}$$

implies that

$$(\pi^*\theta + dd^c u^\epsilon)^{m+1} \wedge \eta \geq (\pi^*\theta + dd^c v)^{m+1} \wedge \eta.$$

In order to simplify notation, in the following we will denote $\pi^*\theta$ by θ and $\theta + dd^c u$ by θ_u , etc.

Setting ϕ to be u^ϵ and ψ to be $v + \epsilon$, we have $\phi = \psi$ near ∂N and $\phi \geq \psi$ on N . We will show that $\phi = \psi$, which implies the theorem by taking $\epsilon \rightarrow 0$.

Set $\rho = \phi - \psi$, so

$$0 \leq \theta_\phi^{m+1} - \theta_\psi^{m+1} = dd^c \rho \wedge \sum_{j=0}^m \theta_\phi^j \wedge \theta_\psi^{m-j}.$$

By Proposition 2.5.1

$$\begin{aligned} 0 &\leq \int_M dd^c \rho \wedge \sum_{j=0}^m \theta_\phi^j \wedge \theta_\psi^{m-j} \wedge \eta \\ &= - \int_M d\rho \wedge d^c \rho \wedge \sum_{j=0}^m \theta_\phi^j \wedge \theta_\psi^{m-j} \wedge \eta, \end{aligned}$$

which implies that

$$(37) \quad d\rho \wedge d^c \rho \wedge \theta_\phi^j \wedge \theta_\psi^{m-j} \wedge \eta = 0,$$

for $j = 0, \dots, m$. We will prove that

$$d\rho \wedge d^c \rho \wedge \theta^m \wedge \eta = 0.$$

We will prove inductively in $k = 0, \dots, m$ that

$$(38) \quad d\rho \wedge d^c \rho \wedge \theta_\phi^i \wedge \theta_\psi^j \wedge \eta = 0, \text{ for } i + j = m - k.$$

This holds for $k = 0$. Assume that it holds for $0, \dots, k-1$.

$$\begin{aligned}\theta_\phi^i \wedge \theta_\psi^j \wedge \theta^k &= \theta_\phi^{i+k} \wedge \theta_\psi^j - dd^c \phi \wedge \alpha, \\ \alpha &= \theta_\phi^i \wedge \theta_\psi^j \wedge \sum_{\ell=0}^{k-1} \theta_\phi^\ell \wedge \theta^{k-1-\ell}.\end{aligned}$$

Thus we have

$$\begin{aligned}d\rho \wedge d^c \rho \wedge \theta_\phi^i \wedge \theta_\psi^j \wedge \theta^k &\leq d\rho \wedge d^c \rho \wedge (T - dd^c \phi \wedge \alpha) \\ &= d(\rho d^c \rho \wedge T - d^c \phi \wedge \alpha d\rho \wedge d^c \rho) \\ &\quad - \rho dd^c \rho \wedge T - d\rho \wedge d^c \phi \wedge \alpha \wedge dd^c \rho,\end{aligned}$$

where $T = \sum_{j=0}^m \theta_\phi^j \wedge \theta_\psi^{m-j}$. From (37) and Proposition 2.5.1

$$(39) \quad \int_N d\rho \wedge d^c \rho \wedge \theta_\phi^i \wedge \theta_\psi^j \wedge \theta^k \wedge \eta \leq - \int_N d\rho \wedge d^c \phi \wedge \alpha \wedge dd^c \rho \eta.$$

But we can bound the right-hand-side by

$$(40) \quad - \int_N d\rho \wedge d^c \phi \wedge \alpha \wedge dd^c \rho \eta \leq \left| \int_N d\rho \wedge d^c \phi \wedge \alpha \wedge \theta_\phi \wedge \eta \right| + \left| \int_N d\rho \wedge d^c \phi \wedge \alpha \wedge \theta_\psi \right|.$$

By the Schwartz inequality

$$\begin{aligned}&\left| \int_N d\rho \wedge d^c \phi \wedge \alpha \wedge \theta_\phi \wedge \eta \right| \\ &\leq \left(\int_N d\rho \wedge d^c \rho \wedge \alpha \wedge \theta_\phi \wedge \eta \right)^{\frac{1}{2}} \left(\int_N d\phi \wedge d^c \phi \wedge \alpha \wedge \theta_\phi \wedge \eta \right)^{\frac{1}{2}}\end{aligned}$$

Since $d\phi \wedge d^c \phi \wedge \alpha \wedge \theta_\phi = 0$ by induction,

$$\int_N d\rho \wedge d^c \phi \wedge \alpha \wedge \theta_\phi \wedge \eta = 0.$$

Similarly,

$$\int_N d\rho \wedge d^c \phi \wedge \alpha \wedge \theta_\psi \wedge \eta = 0.$$

Thus from (39) and (40) we have

$$d\rho \wedge d^c \rho \wedge \theta_\phi^i \wedge \theta_\psi^j \wedge \theta^k \wedge \eta = 0,$$

since it is positive. \square

3. ENERGY FUNCTIONALS

We will define important functionals on the space of potentials \mathcal{H} and consider their extensions to potentials of weak regularity. Fix a Sasakian structure (η, ξ, Φ, g) on M .

3.1. Monge-Ampère energy. We define the *Monge Ampère energy*

$$(41) \quad \mathcal{E}(u) = \frac{1}{m+1} \sum_{j=0}^m \int_M u(\omega_u^T)^j \wedge (\omega^T)^{m-j} \wedge \eta.$$

We will show that this is the same as the Monge Ampère energy defined in (29) for $u \in \mathcal{H}$. But the definition in (41) extends to $u \in \text{PSH}(M, \omega^T) \cap L^\infty(M)$. Let α be a basic, closed, $(1, 1)$ -form on M . We define the α -energy to be

$$(42) \quad \mathcal{E}^\alpha(u) = \sum_{j=0}^{m-1} \int_M u(\omega_u^T)^j \wedge (\omega^T)^{m-j-1} \wedge \alpha \wedge \eta,$$

which is also defined for any $u \in \text{PSH}(M, \omega^T) \cap L^\infty(M)$.

Proposition 3.1.1. *Given $u_1, u_2 \in \text{PSH}(M, \omega^T) \cap C^0(M)$, then*

$$(43) \quad \frac{d}{dt} \mathcal{E}((1-t)u_1 + tu_2)|_{t=0+} = \int_M (u_2 - u_1)(\omega^T + dd^c u_1)^m \wedge \eta,$$

$$(44) \quad \frac{d}{dt} \mathcal{E}^\alpha((1-t)u_1 + tu_2)|_{t=0+} = m \int_M (u_2 - u_1)(\omega^T + dd^c u_1)^{m-1} \wedge \alpha \wedge \eta.$$

Proof. Let $w = u_2 - u_1$. Then

$$\begin{aligned} \mathcal{E}((1-t)u_1 + tu_2) &= \sum_{j=0}^m \int_M (u_1 + tw)((1-t)\omega_{u_1}^T + t\omega_{u_2}^T)^j \wedge (\omega^T)^{m-j} \wedge \eta \\ &= \sum_{j=0}^m \int_M u_1(\omega_{u_1}^T)^j \wedge (\omega^T)^{m-j} \wedge \eta \\ &\quad + t \sum_{j=0}^m \int_M w(\omega_{u_1}^T)^j \wedge (\omega^T)^{m-j} \wedge \eta \\ &\quad + t \sum_{j=1}^m \int_M u_1 j(\omega_{u_1}^T)^{j-1} \wedge dd^c w \wedge (\omega^T)^{m-j} \wedge \eta + O(t^2) \end{aligned}$$

Then by Proposition 2.3.2

$$\begin{aligned} (45) \quad &\sum_{j=1}^m \int_M u_1 j(\omega_{u_1}^T)^{j-1} \wedge dd^c w \wedge (\omega^T)^{m-j} \wedge \eta \\ &= \sum_{j=1}^m \int_M w dd^c u_1 j(\omega_{u_1}^T)^{j-1} \wedge (\omega^T)^{m-j} \wedge \eta \\ &= \sum_{j=1}^m \int_M w j(\omega_{u_1}^T)^j \wedge (\omega^T)^{m-j} \wedge \eta - \sum_{j=0}^{m-1} \int_M w(j+1)(\omega_{u_1}^T)^j \wedge (\omega^T)^{m-j} \wedge \eta. \end{aligned}$$

But

$$\begin{aligned} &\sum_{j=0}^m (\omega_{u_1}^T)^j \wedge (\omega^T)^{m-j} + \sum_{j=1}^m j(\omega_{u_1}^T)^j \wedge (\omega^T)^{m-j} - \sum_{j=0}^{m-1} (j+1)(\omega_{u_1}^T)^j \wedge (\omega^T)^{m-j} \\ &= (m+1)(\omega_{u_1}^T)^m. \end{aligned}$$

Therefore

$$\mathcal{E}((1-t)u_1 + tu_2) = \mathcal{E}(u_1) + t \int_M w(\omega_{u_1}^T)^m \wedge \eta + O(t^2),$$

and the first equation follows. The proof of the second equation is completely analogous. \square

Corollary 3.1.2. *For $u, v \in \text{PSH}(M, \omega^T) \cap C^0(M)$, we have*

$$(46) \quad \mathcal{E}(u) - \mathcal{E}(v) = \frac{1}{m+1} \sum_{j=0}^m \int_M (u-v)(\omega_u^T)^j \wedge (\omega_v^T)^{m-j} \wedge \eta,$$

$$(47) \quad \mathcal{E}^\alpha(u) - \mathcal{E}^\alpha(v) = \sum_{j=0}^{m-1} \int_M (u-v)(\omega_u^T)^j \wedge (\omega_v^T)^{m-j-1} \wedge \alpha \wedge \eta.$$

Note that this implies that both functionals are continuous along C^0 paths in $\text{PSH}(M, \omega^T) \cap C^0(M)$.

Proof. Consider the first equation. Fix v and denote the right-hand expression by $\mathcal{F}(u)$. Applying Proposition 3.1.1 with ω_v^T in place of ω^T gives

$$\frac{d}{dt} \mathcal{F}((1-t)v + tu) = \int_M (u-v)(\omega_{(1-t)v+tu}^T)^m \wedge \eta = \frac{d}{dt} \mathcal{E}((1-t)v + tu).$$

And since both $\mathcal{F}(\cdot)$ and $\mathcal{E}(\cdot) - \mathcal{E}(v)$, both vanish at v the result follows. The second equation follows from a completely analogous argument. \square

Corollary 3.1.3. *Suppose $\{u_t\}$ is a continuous path in $\text{PSH}(M, \omega^T) \cap C^0(M)$, meaning that $u \in C^0([a, b] \times M)$ and $u_t \in \text{PSH}(M, \omega^T) \cap C^0(M)$ for $t \in [a, b]$, and which also has $\dot{u} \in C^0([a, b] \times M)$. Then*

$$(48) \quad \frac{d}{dt} \mathcal{E}(u_t) = \int_M \dot{u}_t (\omega^T + dd^c u_t)^m \wedge \eta,$$

$$(49) \quad \frac{d}{dt} \mathcal{E}^\alpha(u_t) = m \int_M \dot{u}_t (\omega^T + dd^c u_t)^{m-1} \wedge \alpha \wedge \eta.$$

Proof. From Corollary 3.1.2 we have for fixed t

$$\frac{1}{h} (\mathcal{E}(u_{t+h}) - \mathcal{E}(u_t)) = \frac{1}{m+1} \sum_{j=0}^m \int_M \frac{(u_{t+h} - u_t)}{h} (\omega_{u_{t+h}}^T)^j \wedge (\omega_{u_t}^T)^{m-j} \wedge \eta$$

By assumption $\frac{(u_{t+h} - u_t)}{h}$ converges uniformly to \dot{u}_t , thus the formula follows from the weak converges of measures given in Theorem 2.3.1. The same argument gives the second formula. \square

We consider the second variation.

Proposition 3.1.4. *Let $U \in \text{PSH}(N, \pi^* \omega^T) \cap C^0(N)$, that is, a subgeodesic if U is S^1 invariant. Then*

$$(50) \quad d_\tau d_\tau^c \mathcal{E}(u_\tau) = \frac{1}{m+1} \int_M (\pi^* \omega^T + dd^c U)^{m+1} \wedge \eta,$$

$$(51) \quad d_\tau d_\tau^c \mathcal{E}^\alpha(u_\tau) = \int_M (\pi^* \omega^T + dd^c U)^m \wedge \alpha \wedge \eta,$$

where the integration on the right-hand-side denotes the push-forward of currents under the projection $M \times A \rightarrow A$, which is the fiberwise integral if the integrand is sufficiently regular.

Proof. If $\sigma : N = M \times A \rightarrow A$ is the projection, as a current on A we have

$$\mathcal{E}(u_\tau) = \sigma_* \left(U \sum_{j=0}^m (\pi^* \omega^T + dd^c U)^j \wedge (\pi^* \omega^T)^{m-j} \wedge \eta \right).$$

Then

$$\begin{aligned} d_\tau d_\tau^c \mathcal{E}(u_\tau) &= \sigma_* \left(dd^c U \wedge \sum_{j=0}^m (\pi^* \omega^T + dd^c U)^j \wedge (\pi^* \omega^T)^{m-j} \wedge \eta \right) \\ &= \sigma_* \left((\pi^* \omega^T + dd^c U)^{m+1} - (\pi^* \omega^T)^{m+1} \right) \\ &= \sigma_* \left((\pi^* \omega^T + dd^c U)^{m+1} \right). \end{aligned}$$

And a similar argument proves the second formula. \square

Note that if $U \in \text{PSH}(N, \pi^* \omega^T) \cap C_w^{1,1}(N)$, then the integrands in the proposition are differential forms with L^∞ coefficients. Thus the push-forward is ordinary integration along the fibers.

3.2. Mabuchi K-energy. The Mabuchi K-energy is a functional is indispensable in Kähler geometry, as its critical points are constant scalar curvature metrics. It has been defined on Sasakian manifolds also [28]. The *Mabuchi K-energy* is the functional $\mathcal{M} : \mathcal{H} \rightarrow \mathbb{R}$ with derivative

$$(52) \quad d\mathcal{M}|_\phi(\psi) = - \int_M \psi \left(m \text{Ric}_{\omega_\phi^T} \wedge (\omega_\phi^T)^{m-1} - \overline{S}^T (\omega_\phi^T)^m \right) \wedge \eta,$$

where the average

$$\begin{aligned} \overline{S}^T &= \frac{\int_M S_g^T d\mu_g}{\int_M d\mu_g} \\ &= \frac{2m\pi c_1(\mathcal{F}_\xi) \cup [\omega^T]^{m-1}}{[\omega^T]^m} \end{aligned}$$

is independent of the transversal metric.

Consider the transverse canonical line bundle $\Lambda_b^{m,0}$, whose fiber in any foliation chart U_α as in (11) is the line spanned by $dz_\alpha^1 \wedge \dots \wedge dz_\alpha^m$. The transversal Kähler metric ω^T induces a metric on $\Lambda_b^{m,0}$. The metric denoted in each chart by $e^{-\Psi_\alpha}$, i.e. its value on the canonical section. Thus

$$\Psi_\alpha = \log \left(\frac{(\omega^T)^m}{idz_\alpha^1 \wedge d\bar{z}_\alpha^1 \wedge \dots \wedge idz_\alpha^m \wedge d\bar{z}_\alpha^m} \right).$$

Then $-\Psi_\alpha$ is the metric on the transversal anti-canonical bundle, or rather its weight, while the metric in local coordinates is e^{Ψ_α} . More generally, given any measure μ absolutely continuous w.r.t. $d\mu_\eta = (\omega^T)^m \wedge \eta$, and invariant under the Reeb flow, we have $\Psi_\alpha^\mu = \log(f) + \Psi_\alpha$, where $f = \frac{d\mu}{d\mu_\eta}$ is the Radon-Nikodym derivative, and $e^{\Psi_\alpha^\mu}$ defines a singular metric on the transversal anti-canonical bundle.

We recall the entropy of a measure. If μ is absolutely continuous with respect to μ_0 then the *entropy* of μ relative to μ_0 is

$$(53) \quad H_{\mu_0}(\mu) := \int_M \log\left(\frac{d\mu}{d\mu_0}\right) d\mu.$$

In the sequel we will assume μ_0 the measure induced by $d\mu_\eta = (\omega^T)^m \wedge \eta$.

We give an alternative formula for the Mabuchi energy, first given in [19] in the Kähler case.

$$(54) \quad \mathcal{M}(u) = (\bar{S}^T \mathcal{E}(u) - \mathcal{E}^{\text{Ric}_{\omega^T}}(u)) + H_{\mu_0}(d\mu_u),$$

where $d\mu_u = (\omega_u^T)^m \wedge \eta$. The functional \mathcal{M} is easily seen to be extended to $\mathcal{H}_{1,1} = \text{PSH}(M, \omega^T) \cap C_w^{1,1}(M)$. One can check that this definition coincides with the above on \mathcal{H} , by differentiating (54) along smooth paths, which give the formula in (52).

3.3. Convexity of the Mabuchi K-energy. We prove Theorem 1 in this section, that $\mathcal{M}(u_t)$ is convex along a weak $C_w^{1,1}$ geodesic. First we prove that as a current on A , $\mathcal{M}(u_\tau)$ is positive. The proof follows the same basic approach as that of R. Berman and B. Berndtsson [41] in the Kähler case, but because one has to be careful when considering currents and weak differentiation in the transversal holomorphic foliation situation, we give a complete proof. The proof is based on properties of local Bergman kernels.

We recall that given a complex manifold X , not assumed to be compact, with a line bundle L and a bounded metric ϕ , the Bergman kernel $K_{k\phi}$ is the restriction to the diagonal of the section of $(kL + K_X) \otimes \overline{(kL + K_X)}$ giving the reproducing kernel. For the L^2 norm we take

$$\|\sigma\|_{L^2} = \int_X \sigma \wedge \bar{\sigma} e^{-k\phi},$$

where $\sigma \in \Gamma(kL + K_X)$. Contracting by the metric on kL gives a measure

$$(55) \quad \beta_k := \frac{m!}{k^m} K_{k\phi} e^{-k\phi},$$

where $\dim X = m$.

The following convergence result is used. See [41] for the proof.

Theorem 3.3.1 ([41]). *Let $\mathbb{B} \subset \mathbb{C}^m$ be the unit ball, and let ϕ be a plurisubharmonic function defined on a neighborhood of \mathbb{B} and in $C_w^{1,1}$. If $d\mu$ denotes Lebesgue measure, then on any relatively compact subdomain $E \subset \mathbb{B}$ we have $\beta_k \leq C_E d\mu$. And after passing to a subsequence*

$$\lim_{k \rightarrow \infty} \beta_k(x) = (dd^c \phi)^m(x),$$

for almost all $x \in \mathbb{B}$. Thus $\beta_k \rightarrow (dd^c \phi)^m$ in total variation norm.

In the following theorem $D \subset \mathbb{C}$ is only required to be a smoothly bounded domain.

Theorem 3.3.2. *Let $\{u_\tau\}$ be a $C_w^{1,1}$ solution to the transversal Monge-Ampère equation on $N = M \times D$, that is if $U = \{u_\tau\}$, then $U \in \text{PSH}(N, \pi^* \omega^T) \cap C_w^{1,1}(N)$ and $(\pi^* \omega^T + dd^c U)^{m+1} \wedge \eta = 0$. Then the Mabuchi functional $\mathcal{M}(u_\tau)$ is weakly subharmonic with respect to $\tau \in D$. In particular, if $\{u_t\}$ is a weak geodesic connecting two elements of \mathcal{H} then $\mathcal{M}(u_t)$ is weakly convex.*

Proof. Let $\Psi = \Psi(\tau, x) = \psi_\tau(x)$ be a possibly singular bounded metric on $\Lambda_b^{m,0} = K_{\mathcal{F}_\xi}$. We define

$$(56) \quad f^\Psi(\tau) := (\overline{S}^T \mathcal{E}(u) - \mathcal{E}^{\text{Ric}_{\omega^T}}(u)) + \int_M \log\left(\frac{e^{\psi_\tau}}{(\omega^T)^m}\right) (\omega_{u_\tau}^T)^m \wedge \eta.$$

First we claim that as a current on D

$$(57) \quad dd^c f^\Psi(\tau) = \int_M T, \quad T = dd^c(\Psi(\pi^* \omega^T + dd^c U)^m) \wedge \eta,$$

where T is a transversal $(m+1, m+1)$ current on N . Part of the problem is to show that T is positive and thus defines a measure on N . If $v \in C_0^\infty(D)$ and $\varpi : N \rightarrow D$ is the projection, then by (57) we mean

$$\langle dd^c f^\Psi, v \rangle := \int_N \Psi(\pi^* \omega^T + dd^c U)^m \wedge dd^c(\varpi^* v) \wedge \eta.$$

Let Ψ_j be a sequence of uniformly bounded smooth metrics $\Psi_j \rightarrow \Psi$ almost everywhere on M for each $\tau \in D$. This sequence can be constructed by taking convolutions in each foliation chart as in (31) with a subordinate partition of unity. In order to get basic metrics Ψ_j one must average by the torus $T \subset \text{Aut}(\eta, \xi, \Phi, g)$ generated by ξ .

Making use of Proposition 3.1.4 we compute

$$dd^c f^{\Psi_j}(\tau) = \int_M T_j, \quad T_j = dd^c(\Psi_j(\pi^* \omega^T + dd^c U)^m) \wedge \eta,$$

and since $\int_M T_j \rightarrow \int_M T$ weakly as currents by Lebesgue dominated convergence, and likewise $f^{\Psi_j} \rightarrow f^\Psi$ pointwise, we have (57).

Let $\Theta_j = dd^c \Psi_j$. We extend T to be a transversal current on N by

$$\langle T, w \rangle := \lim_{j \rightarrow \infty} \int_N w \Theta_j \wedge (\pi^* \omega^T + dd^c U)^m \wedge \eta, \quad \text{for } w \in C_0^\infty(N).$$

This is *a priori* not defined for all test functions. The proof will proceed by showing when Ψ is the metric induce by $(\omega_{u_\tau}^T)^m$ that in each holomorphic foliation chart

$$\lim_{j \rightarrow \infty} \Theta_j \wedge (\pi^* \omega^T + dd^c U)^m = dd^c(\Psi(\pi^* \omega^T + dd^c U)^m)$$

is a positive current, and so T defines a positive Radon measure on N .

We want to consider the singular metric $\psi_\tau = \log(\pi^* \omega^T + dd^c u_\tau)^m$, but an additional problem is that it is not locally bounded. Fix $A > 0$ and define

$$\Psi_A = \max\{\log(\pi^* \omega^T + dd^c u_\tau)^m, \chi - A\},$$

where χ is a fixed continuous metric on $\Lambda_b^{m,0}$ that we will define. We will prove that the transversal $(m+1, m+1)$ current

$$(58) \quad T_A = dd^c \Psi_A \wedge (\pi^* \omega^T + dd^c U)^m,$$

satisfies $T_A \geq 0$ if χ is chosen so that

$$(59) \quad dd^c \chi \geq -k_0(\pi^* \omega^T + dd^c U),$$

for some $k_0 \in \mathbb{N}$. This implies that $dd^c f^{\Psi_A} \geq 0$ on D . But dominated convergence implies $f^{\Psi_A}(\tau) \rightarrow \mathcal{M}(u_\tau)$ pointwise, thus

$$dd^c f^{\Psi_A}(\tau) \rightarrow dd^c \mathcal{M}(u_\tau) \quad \text{weakly}$$

and $dd^c \mathcal{M}(u_\tau) \geq 0$ as a current.

In order to satisfy (59) first let χ_0 be an arbitrary smooth metric on $\Lambda_b^{m,0}$, and choose $k_0 \in \mathbb{N}$ large enough that $k_0\omega^T + dd^c\chi_0 \geq 0$. Set $\chi = \pi^*\chi_0 - k_0U$, then

$$\begin{aligned} dd^c\chi &= \pi^*dd^c\chi_0 - k_0(\pi^*\omega^T + dd^cU) + k_0\pi^*\omega^T \\ &\geq -k_0(\pi^*\omega^T + dd^cU). \end{aligned}$$

We next prove that $T_A \geq 0$ in local foliation charts. This will follow from a local approximation of the metric Ψ_A by Bergman densities. Consider a fixed transversal holomorphic chart

$$\phi_\alpha : U_\alpha \times V \rightarrow W_\alpha \times V, \quad U_\alpha \times V \subset M \times D.$$

Write

$$\pi^*\omega^T + dd^cU = dd^c\Psi$$

for a plurisubharmonic function on $W_\alpha \times V$. We will denote $\phi_\tau = \Phi(\cdot; \tau)$.

Let β_k be the Bergman measure for the Hilbert space of holomorphic functions on the unit ball $\mathbb{B} \subset W_\alpha \subset \mathbb{C}^m$ with weight $k\phi_\tau$. Consider the transversal current

$$T_{A,k} := dd^c\Psi_{A,k} \wedge (dd^c\Phi)^m, \quad \Psi_{A,k} := \max\{\log \beta_{k\phi_\tau}, \chi - A\}.$$

By Theorem 3.3.1 and dominated convergence

$$(60) \quad \lim_{k \rightarrow \infty} T_{A,k} = T_A \quad \text{weakly}$$

as transversal currents. By a result of B. Berndtsson [5] on the plurisubharmonic variation of Bergman kernels, $dd^c K_{k\phi_\tau} \geq 0$ on $B \times V$. Thus

$$dd^c \log \beta_{k\phi_\tau} \geq -k dd^c \Phi.$$

Since by (59) we have $dd^c\chi \geq -k_0 dd^c\Phi$, for $k \geq k_0$

$$dd^c\Psi_{A,k} \geq -k dd^c\Phi.$$

Thus

$$T_{A,k} = dd^c\Psi_{A,k} \wedge (dd^c\Phi)^m \geq -k(dd^c\Phi)^{m+1} = 0.$$

And finally, by Theorem 3.3.1

$$e^{\Psi_{A,k}} = \max\left\{\frac{m!}{k^m} K_{k\phi_\tau} e^{-k\phi_\tau}, e^{-(\chi-A)}\right\} \rightarrow \left\{(dd^c\phi_t au)^m, e^{-(\chi-A)}\right\}$$

pointwise almost everywhere on M for all $\tau \in D$ in a dominated fashion. Thus (60) holds, and the proof is complete. \square

It remains to show that when $\{u_\tau\}$ is a weak geodesic, so $D = A = \{\tau \in \mathbb{C} \mid 1 \leq |\tau| \leq e\}$ and u_τ depends only on $|\tau| = e^t$, that $\mathcal{M}(u_t)$ is convex in the pointwise sense and thus continuous.

Theorem 3.3.3. *Suppose that u_t is a weak $C_w^{1,1}$ geodesic. Then $\mathcal{M}(u_t)$ is continuous and therefore pointwise convex.*

Proof. Let Ψ_A be defined as above. We will prove that f^{Ψ_A} is convex and continuous. Then f^Ψ will be convex by taking $A \rightarrow \infty$.

Let $\kappa_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$, for $\epsilon > 0$, be a sequence of strictly convex functions with $\kappa'_\epsilon \geq 0$ tending to Id as $\epsilon \rightarrow 0$. Define $f_\epsilon^{\Psi_A}$ just as f^{Ψ_A} but with

$$\log\left(\frac{e^{\psi_{A,\tau}}}{(\omega^T)^m}\right)$$

replaced by

$$\kappa_\epsilon \left(\log \left(\frac{e^{\psi_{A,\tau}}}{(\omega^T)^m} \right) \right).$$

We will prove that $f_\epsilon^{\Psi_A}$ is convex for all $\epsilon > 0$. Note that by the argument in the last theorem $f_\epsilon^{\Psi_A}$ is weakly convex. Let $\{\sigma_\alpha\}$ be a partition of unity subordinate to the covering of M by foliation charts $\{U_\alpha\}$. And consider the local entropy functions

$$H_\alpha = \int_M \sigma_\alpha \kappa_\epsilon \left(\log \left(\frac{e^{\psi_{A,\tau}}}{(\omega^T)^m} \right) \right) (\omega_{\phi_\tau}^T)^m \wedge \eta$$

$$\int_{W_\alpha} \hat{\sigma}_\alpha \kappa_\epsilon \left(\log \left(\frac{e^{\psi_{A,\tau}}}{(\omega^T)^m} \right) \right) (\omega_{\phi_\tau}^T)^m$$

where $\hat{\sigma}_\alpha$ is σ_α integrated along the leaves. Define H_α^k by replacing Ψ_A by its approximation by local Bergman kernels

$$\Psi_{A,k} = \max\{\log \beta_{k\phi_\tau}, \chi - A\}.$$

We take $d_\tau d_\tau^c$ of H_α^k ,

$$d_\tau d_\tau^c H_\alpha^k = \int_{W_\alpha} dd^c \left(\hat{\sigma}_\alpha \kappa_\epsilon \left(\log \left(\frac{e^{\psi_{A,\tau}}}{(\omega^T)^m} \right) \right) \right) (\omega_{\phi_\tau}^T)^m,$$

and use the plurisubharmonic variation of Bergman kernels [5], the strict convexity of κ_ϵ , and the fact that dd^c commutes with the push-forward $W_\alpha \times A \rightarrow A$ to get

$$(61) \quad d_\tau d_\tau^c H_\alpha^k \geq -C_{\epsilon,\alpha}.$$

Thus $H_\alpha^k + C_{\epsilon,\alpha}t^2$ is convex since H_α^k is continuous. Taking $k \rightarrow \infty$ we get that $H_\alpha + C_{\epsilon,\alpha}t^2$ is convex by dominated convergence. Summing over α gives that

$$\int_M \kappa_\epsilon \left(\log \left(\frac{e^{\psi_{A,\tau}}}{(\omega^T)^m} \right) \right) (\omega_{\phi_\tau}^T)^m \wedge \eta + C_\epsilon t^2$$

is convex and thus continuous. Therefore $f_\epsilon^{\Psi_A}$ is continuous and poinwise convex, and f^{Ψ_A} is convex by dominated convergence. \square

We now prove Corollary 3.

Lemma 3.3.4. *Let $\{\phi_t\}$ be the weak $C_w^{1,1}$ geodesic connecting $\phi_0, \phi_1 \in \mathcal{H}$. Then*

$$\frac{d}{dt} \mathcal{M}(\phi_t)|_{t=0+} \geq - \int_M (S_{\phi_0}^T - \overline{S}) \frac{d\phi_t}{dt}|_{t=0+} (\omega_{\phi_0}^T)^m \wedge \eta.$$

Proof. Let $\mu = (\omega^T)^m \wedge \eta$. H_μ is convex on measure with volume $\text{Vol}(\mu)$. Set $\nu_0 = (\omega_{\phi_0}^T)^m \wedge \eta$ and $\nu_1 = (\omega_{\phi_1}^T)^m \wedge \eta$. Hence if $\nu_s = s\nu_1 + (1-s)\nu_0$, we have

$$H_\mu(\nu_1) - H_\mu(\nu_0) \geq \frac{d}{ds} H_\mu(\nu_s)|_{s=0} = \int_M \log\left(\frac{\nu_0}{\mu}\right) (d\nu_1 - d\nu_0).$$

Thus

(62)

$$\begin{aligned}
& \frac{1}{t} (H_\mu((\omega_{\phi_t}^T)^m \wedge \eta) - H_\mu((\omega_{\phi_0}^T)^m \wedge \eta)) \\
& \geq \int_M \log\left(\frac{(\omega_{\phi_0}^T)^m \wedge \eta}{(\omega_{\phi_t}^T)^m \wedge \eta}\right) \frac{1}{t} ((\omega_{\phi_t}^T)^m - (\omega_{\phi_0}^T)^m) \wedge \eta \\
& = \int_M \log\left(\frac{\nu_0}{\mu}\right) \frac{1}{t} dd^c(\phi_t - \phi_0) \wedge \sum_{j=0}^{m-1} (\omega_{\phi_t}^T)^j \wedge (\omega_{\phi_0}^T)^{m-j-1} \wedge \eta \\
& = \int_M dd^c \log\left(\frac{\nu_0}{\mu}\right) \frac{1}{t} (\phi_t - \phi_0) \wedge \sum_{j=0}^{m-1} (\omega_{\phi_t}^T)^j \wedge (\omega_{\phi_0}^T)^{m-j-1} \wedge \eta.
\end{aligned}$$

We take the limit of t to zero and apply Theorem 2.3.1

$$\lim_{t \rightarrow 0^+} \frac{1}{t} (H_\mu((\omega_{\phi_t}^T)^m \wedge \eta) \geq m \int_M \frac{d\phi_t}{dt} \Big|_{t=0^+} (\text{Ric}_{\omega_{\phi_t}^T} - \text{Ric}_{\omega_{\phi_0}^T}) \wedge (\omega_{\phi_0}^T)^{m-1} \wedge \eta.$$

Then the *energy* part of (54) differentiates as in Corollary 3.1.3 completing the proof. \square

We now can prove Corollary 3. By the sub-slope property of convex functions

$$\mathcal{M}(\phi_1) - \mathcal{M}(\phi_0) \geq \frac{d}{dt} \mathcal{M}(\phi_t) \Big|_{t=0^+}.$$

Thus

$$\begin{aligned}
\mathcal{M}(\phi_1) - \mathcal{M}(\phi_0) & \geq - \int_M (S_{\phi_0}^T - \overline{S}) \frac{d\phi_t}{dt} \Big|_{t=0^+} (\omega_{\phi_0}^T)^m \wedge \eta, \\
& \geq - \left(\text{Cal}(\phi_0) \right)^{\frac{1}{2}} \left(\int_M \left(\frac{d\phi_t}{dt} \Big|_{t=0^+} \right)^2 d\mu_{\phi_0} \right)^{\frac{1}{2}}
\end{aligned}$$

by the Cauchy-Schwartz inequality. But the distance $d(\phi_0, \phi_1) = \left(\int_M \left(\frac{d\phi_t}{dt} \Big|_{t=0^+} \right)^2 d\mu_{\phi_0} \right)^{\frac{1}{2}}$.

4. UNIQUENESS OF CONSTANT SCALAR CURVATURE STRUCTURES AND EXTREMAL STRUCTURES

4.1. Automorphism groups. We will need some facts about the Lie algebra $\mathfrak{hol}^T(\xi, J)$ and automorphism groups of $C(M)$ when $\mathcal{S}(\xi, J)$ admits an extremal structure. The following was proved in [13], and [45] giving the last statement.

Theorem 4.1.1. *Let $(\eta, \xi, \Phi, g) \in \mathcal{S}(\xi, J)$ be a Sasaki-extremal structure. Then we have the semidirect sum decomposition*

$$(63) \quad \mathfrak{hol}^T(\xi, J) = \mathfrak{a} \oplus \mathfrak{hol}^T(\xi, J)_0,$$

where \mathfrak{a} is the Lie algebra of parallel, with respect to g^T , sections of $\nu(\mathcal{F}_\xi)$. And we also have

$$(64) \quad \mathfrak{hol}^T(\xi, J)_0 = \mathfrak{g} \oplus J\mathfrak{g} \oplus \left(\bigoplus_{\lambda > 0} \mathfrak{h}^\lambda \right),$$

where $\mathfrak{g} = \text{aut}(g, \eta, \xi, \Phi) / \mathbb{R}\xi$ is the image under $\partial_g^\#$ of the imaginary valued functions in \mathcal{H}_g and $\mathfrak{h}^\lambda = \{\overline{X} \in \mathfrak{hol}^T(\xi, J)_0 : [\partial_g^\# s_g, \overline{X}] = \lambda \overline{X}\}$ and $\mathfrak{g} \oplus J\mathfrak{g} = C_{\mathfrak{hol}^T(\xi, J)_0}(\partial_g^\# s_g)$, the centralizer of $\partial_g^\# s_g$.

Furthermore, the connected component of the identity $G = \text{Aut}(\eta, \xi, \Phi, g)_0 \subset \text{Fol}(M, \mathcal{F}_\xi, J)$ is a maximal compact connected subgroup. And any other maximal compact connected subgroup is conjugate to G in $\text{Fol}(M, \mathcal{F}_\xi, J)$.

We will need the following definition from [28].

Definition 4.1.2. A complex vector field X on a Sasakian manifold M is Hamiltonian holomorphic if

- (i) X is basic, i.e. on each chart U_α it projects, and $\pi_\alpha(X)$ is holomorphic on V_α .
- (ii) $u_X := \sqrt{-1}\eta(X)$ is a holomorphy potential,

$$\bar{\partial}_b u_X = -\sqrt{-1}X \lrcorner \omega^T.$$

We denote the space of Hamiltonian holomorphic vector fields by $\mathfrak{h}^{\text{Ham}}$.

We list some useful properties.

- Proposition 4.1.3.** (i) The space of $\mathfrak{h}^{\text{Ham}}$ is a Lie algebra, and $u_{[X,Y]} = Xu_Y - Yu_X$ for $X, Y \in \mathfrak{h}^{\text{Ham}}$.
- (ii) There is a Lie algebra isomorphism $\mathfrak{hol}^T(\xi, J)_0 \cong \mathfrak{h}^{\text{Ham}}/\mathbb{C}\xi$.
- (iii) $\mathfrak{h}^{\text{Ham}}$ is isomorphic to the Lie algebra of holomorphic vector fields \tilde{X} on $C(M)$ for which $[\xi, \tilde{X}] = [r\partial_r, \tilde{X}] = 0$.
- It follows that $\mathfrak{h}^{\text{Ham}}$ only depends on ξ and the complex structure on $C(M)$.

Proof. (i) Suppose $X, Y \in \mathfrak{h}^{\text{Ham}}$. Write $X = X_D - \sqrt{-1}u_X\xi$ with $X_D \in \Gamma(D^{1,0})$ and $X_D = \partial^\# u_X$ as a section of $\nu(\mathcal{F}_\xi)$ and $Y = Y_D - \sqrt{-1}u_Y\xi$. Then

$$\begin{aligned} [X, Y] &= [X_D, Y_D] - \sqrt{-1}([u_X\xi, Y_D] + [X_D, u_Y\xi]) \\ &= [X_D, Y_D] - \sqrt{-1}(X_D u_Y - Y_D u_X). \end{aligned}$$

It is easy to check that if we set $u_{[X,Y]} = X_D u_Y - Y_D u_X = Xu_Y - Yu_X$, then $\partial^\# u_{[X,Y]} = [X_D, Y_D]$. And further more $\eta([X_D, Y_D]) = 2\omega^T(X_D, Y, D) = 0$.

(ii) Suppose $V \in \mathfrak{hol}^T(\xi, J)_0$ and $V = \partial^\# u$ with the holomorphy potential u defined up to a constant. Then taking $V \in \Gamma(D^{1,0})$, $X = V - \sqrt{-1}u\xi$ is representative of $\mathfrak{h}^{\text{Ham}}$.

(iii) If $X \in \mathfrak{h}^{\text{Ham}}$, then a direct computation shows that $\tilde{X} = X + u_X r\partial_r$ is holomorphic on $C(M)$.

Conversely, if \tilde{X} is holomorphic and $[\xi, \tilde{X}] = [r\partial_r, \tilde{X}] = 0$, then it can be written as

$$\tilde{X} = X_D - \sqrt{-1}u_X\xi + u_X r\partial_r,$$

with $X_D \in \Gamma(D^{1,0})$ and $\xi u_X = 0$. A computation using the warped product formulas shows that if $V \in \Gamma(D^{0,1})$, then $\nabla_V^{0,1} \tilde{X} = 0$ implies that $\bar{\partial}_b u_X(V) = g^T(X, V)$. Thus $X_D - \sqrt{-1}u_X\xi \in \mathfrak{h}^{\text{Ham}}$. \square

For a given Sasakian structure (η, ξ, Φ, g) we denote by \mathcal{H} the space of *holomorphy potentials*, that is, the space of $h \in C_b^\infty(M)$ with

$$\partial^\# h \in \mathfrak{hol}^T(\xi, J)_0.$$

Note that $h \in \mathcal{H}$ uniquely determines an element of $V_h \in \mathfrak{h}^{\text{Ham}}$, so we will also use $\partial^\# h$ to denote this element $V_h \in \mathfrak{h}^{\text{Ham}}$. From Proposition 4.1.3 we have a Lie algebra isomorphism $\mathcal{H} \cong \mathfrak{h}^{\text{Ham}}$. Thus an element $V \in \mathfrak{h}^{\text{Ham}}$ has a unique

potential h^V , while as an element of $\mathfrak{hol}^T(\xi, J)_0$ the potential of V is unique up to a constant. The next observation is important.

Proposition 4.1.4. *$h \in \mathcal{H}$ corresponds to $V_h \in \mathfrak{h}^{\text{Ham}}$ with $\text{Re } V_h$ preserving η , and thus (η, ξ, Φ, g) , if and only if up to a constant $h \in C_b^\infty(M, \sqrt{-1}\mathbb{R})$.*

We fix some important notation. Let H be the connected group with Lie algebra $\mathfrak{h}^{\text{Ham}}$, and let $G \subset H$ be a maximal connected compact subgroup. We may assume, by taking G -averages, that we have a G -invariant structure (η, ξ, Φ, g) . Let $K \subseteq G$ be connected compact, and let $\mathfrak{k}, \mathfrak{g}$ be their Lie algebras. We also define

- $\mathfrak{z} = Z(\mathfrak{k})$, the center of \mathfrak{k} ,
- $\mathfrak{z}' = C_{\mathfrak{h}^{\text{Ham}}}(\mathfrak{k})$, the centralizer of \mathfrak{k} in $\mathfrak{h}^{\text{Ham}}$,
- $\mathfrak{p} = N_{\mathfrak{h}^{\text{Ham}}}(\mathfrak{k})$, the normalizer of \mathfrak{k} in $\mathfrak{h}^{\text{Ham}}$.

We denote the corresponding space holomorphy potentials to be $\mathcal{H}^{\mathfrak{k}}, \mathcal{H}^{\mathfrak{z}}$, etc. Note that $\mathcal{H}^{\mathfrak{z}}$ (respectively $\mathcal{H}^{\mathfrak{z}'}$) consist of K -invariant potentials in $\mathcal{H}^{\mathfrak{k}}$ (respectively \mathcal{H}).

We have the injection

$$(65) \quad \mathfrak{z}'/\mathfrak{z} \hookrightarrow \mathfrak{p}/\mathfrak{k}.$$

Proposition 4.1.5. *The inclusion (65) is surjective, so we have an isomorphism of Lie algebras*

$$\mathfrak{z}'/\mathfrak{z} \cong \mathfrak{p}/\mathfrak{k}.$$

Proof. We claim $\mathfrak{z}' + \mathfrak{k}$. Let $W \in \mathfrak{h}^{\text{Ham}}$ with $W = \partial^\# h^W$.

$$W \in \mathfrak{p} \Leftrightarrow [X, W] \in \mathfrak{k}, \forall X \in \mathfrak{k}.$$

So $\text{Re } X(h^W) = h^{[\text{Re } X, W]} = \frac{1}{2}h^{[X, W]} \in \mathcal{H}^{\mathfrak{k}}$, since $\text{Re } X$ is contact. Thus for $\gamma \in K$, $\gamma^* h^W - h^W \in \mathcal{H}^{\mathfrak{k}}$. If we average with respect to Haar measure on K , we get a K -invariant \tilde{h}^W so that $\hat{h}^W := h^W - \tilde{h}^W \in \mathcal{H}^{\mathfrak{k}}$. We have

$$W = \partial^\# \tilde{h}^W + \partial^\# \hat{h}^W \in \mathfrak{z}' + \mathfrak{k},$$

because $\partial^\# \tilde{h}^W \in \mathfrak{z}'$ since it is K -invariant. \square

Suppose that (η, ξ, Φ, g) is a Sasakian structure with $G = \text{Aut}(g, \eta, \xi, \Phi)_0 \subset \text{Fol}(M, \mathcal{F}_\xi, J)$ a maximal compact subgroup. This is equivalent to requiring that $G \subset H$ is maximal compact. Here we are taking $K = G$ in the above notation. We have an orthogonal decomposition using the volume form $d\mu_g = (\omega^T)^m \wedge \eta$

$$C_b^\infty(M)^G = \sqrt{-1}\mathcal{H}_g^{\mathfrak{z}} \oplus W_g,$$

with projections

$$\pi^G : C_b^\infty(M)^G \rightarrow \sqrt{-1}\mathcal{H}_g^{\mathfrak{z}} \text{ and } \pi_g^W : C_b^\infty(M)^G \rightarrow W_g.$$

The *extremal vector field* is defined to be

$$(66) \quad V = \partial^\# \pi^G(S_g) \in \mathfrak{hol}^T(\xi, J)_0.$$

One can check that V is independent of the of the G -invariant Sasakian structure in $\mathcal{S}(\xi, J)$. See [27], where the arguments can be applied in the Sasakian case also.

We define the *reduced scalar curvature* to be

$$(67) \quad S_g^G = \pi_g^W(S_g).$$

Note that a G -invariant structure in $\mathcal{S}(\xi, J)$ has vanishing reduced scalar curvature if and only if it is Sasaki-extremal.

Next suppose (η, ξ, Φ, g) is Sasaki-extremal. Then $G = \text{Aut}(\eta, \xi, \Phi, g)_0$ is a maximal compact subgroup of H . Furthermore, if it is cscS, then H is the complexification of G . Let $P := N_H(G)_0$ be the connected component of the identity of the normalizer of G in H . Define $Z' = C_H(G)_0$, the identity component of the centralizer of G in H , and $Z_0 = Z' \cap G$.

Proposition 4.1.6. *Let (η, ξ, Φ, g) be cscS (respectively Sasaki-extremal), then its orbit $\mathcal{O} = H/G$ (respectively its orbit under P , $\mathcal{O}_P = P/G = Z'/Z_0$) is a symmetric space with the Riemannian structure induced by $\mathcal{O} \subset \tilde{\mathcal{H}}$ (respectively $\mathcal{O}_P \subset \tilde{\mathcal{H}}^G$).*

It follows that the exponential map

$$\exp : \mathfrak{h}^{\text{Ham}}/\mathfrak{g} \rightarrow \mathcal{O} \quad (\text{resp. } \exp : \mathfrak{p}/\mathfrak{g} \rightarrow \mathcal{O}_P)$$

is onto.

Proof. Suppose $Y \in \mathfrak{h}^{\text{Ham}}$ and $\partial^\# u^Y = Y$ with $u^Y \in \mathcal{H} \cap C_b^\infty(M, \mathbb{R})$. We first prove

Lemma 4.1.7. *Let $g_t = \exp(t \text{Re } Y) \in H$. Then $\{g_t^* \omega^T \mid t \in \mathbb{R}\}$ is a geodesic in \mathcal{K} .*

We define a path $\{\phi_t \mid t \in \mathbb{R}\} \subset \mathcal{K} \cong \tilde{H}$ by $g_t^* \omega^T = \omega_{\phi_t}^T$.

$$\sqrt{-1} \partial \bar{\partial} \dot{\phi}_t = \frac{d}{dt} \omega_{\phi_t}^T = \mathcal{L}_{\text{Re } Y} \omega_{\phi_t}^T = d(Y \lrcorner \omega_{\phi_t}^T) = \sqrt{-1} \partial \bar{\partial} u_{\omega_{\phi_t}^T}^Y$$

So if $\tilde{u}_{\omega_{\phi_t}^T}^Y$ is normalized so that $\int \tilde{u}_{\omega_{\phi_t}^T}^Y (\omega_{\phi_t}^T)^m \wedge \eta = 0$, then $\dot{\phi}_t = \tilde{u}_{\omega_{\phi_t}^T}^Y$. Thus we have

$$\sqrt{-1} \partial \bar{\partial} \ddot{\phi}_t = \sqrt{-1} \frac{d}{dt} \partial \bar{\partial} \dot{\phi}_t = \frac{d}{dt} \mathcal{L}_{\text{Re } Y} \omega_{\phi_t}^T = \mathcal{L}_{\text{Re } Y} \sqrt{-1} \partial \bar{\partial} \dot{\phi}_t = \sqrt{-1} \partial \bar{\partial} (Y \dot{\phi}_t).$$

Therefore

$$\ddot{\phi}_t - \frac{1}{2} |d\dot{\phi}_t|_{\omega_{\phi_t}^T}^2 = C_t$$

for $C_t \in \mathbb{R}$ for all $t \in \mathbb{R}$. But we have

$$\begin{aligned} 0 &= \frac{d}{dt} \mathcal{E}(\phi_t) = \frac{d}{dt} \langle \dot{\phi}_t, 1 \rangle_{\omega_{\phi_t}^T} \\ &= \left\langle \frac{D}{dt} \dot{\phi}_t, 1 \right\rangle_{\omega_{\phi_t}^T} = C(t) \text{Vol}(M), \end{aligned}$$

thus $C_t = 0$, and the lemma is proved.

One can check that for $g \in H$

$$\mathcal{K} \ni \omega^T \mapsto g^* \omega^T \in \mathcal{K}$$

is an isometry of \mathcal{K} . By Theorem 4.1.1, in the cscS (respectively the Sasaki-extremal) case (H, G) (respectively (Z', Z_0)) is a symmetric pair. Proposition 4.1.5 shows that in the Sasaki-extremal case the orbit $\mathcal{O} \cong \mathfrak{p}/\mathfrak{g}$ is isomorphic to Z'/Z_0 . The inclusions

$$\mathcal{O} \subset \tilde{\mathcal{H}} \quad (\text{respectively } \mathcal{O}_P \subset \tilde{\mathcal{H}}^G)$$

induce a homogeneous Riemannian structure on \mathcal{O} (respectively \mathcal{O}_P). Then in both cases, $\mathcal{O} \cong H/G$ ($\mathcal{O}_P \cong Z'/Z_0$) has the structure of a Riemannian symmetric space with the induced metric [30, Prop. 3.4]. \square

4.2. Modified Mabuchi functional. We will consider several modification to the Mabuchi functional. The first is useful because the Mabuchi energy \mathcal{M} is not known to be strictly convex on weak geodesics. The other cases give Mabuchi functionals characterizing constant α -twisted scalar curvature and Sasaki-extremal structures.

Let μ be a smooth, strictly positive volume form on M , which for simplicity we assume to be invariant of the Reeb flow. We define

$$(68) \quad \mathcal{F}^\mu(u) := \int_M u \, d\mu - c_\mu \mathcal{E}(u),$$

where $c_\mu > 0$ is chosen so that $\mathcal{F}^\mu(1) = 0$. Denote $J_\mu(u) = \int_M u \, d\mu$.

Proposition 4.2.1. *\mathcal{F}^μ is strictly convex along weak $C_w^{1,1}$ geodesics. In fact, if $\{u_t\}$ is a weak geodesic $J_\mu(u_t)$ is strictly convex, in that if $J_\mu(u_t)$ is affine then $\omega_{u_t}^T$ is constant.*

More precisely, if $\omega_{u_t}^T \leq C\omega^T$ and $\nu \geq A(\omega^T)^m \wedge \eta$, then

$$\frac{d}{dt} J_\mu(u_t)|_{t=b} - \frac{d}{dt} J_\mu(u_t)|_{t=a} \geq \hat{C} d(\omega_{u_a}^T, \omega_{u_b}^T)^2,$$

$b > a$, where $\hat{C} > 0$ depends only on C, μ, ω^T , and M .

Proof. First suppose that $\{u_t\}$ is a smooth subgeodesic, thus $\ddot{u}_t \geq \frac{1}{2} |d\dot{u}_t|_{\omega_{u_t}^T}^2$. Suppose $\omega_{u_t}^T \leq C\omega^T$ and $\nu \geq A(\omega^T)^m \wedge \eta$, then

$$\begin{aligned} \frac{d^2}{dt^2} J_\mu(u_t) &= \int_M \ddot{u}_t \, d\mu \geq \int_M |\bar{\partial}\dot{u}_t|_{\omega_{u_t}^T}^2 \, d\mu \\ &\geq C^{-1} \int_M |\bar{\partial}\dot{u}_t|_{\omega^T}^2 \, d\mu \\ &\geq \frac{A}{c} \int_M |\bar{\partial}\dot{u}_t|_{\omega^T}^2 (\omega^T)^m \wedge \eta \\ &\geq \frac{A}{c} \tilde{C} \int_M |\dot{u}_t - c_t|^2 (\omega^T)^m \wedge \eta, \end{aligned}$$

where the last step in the Poincaré inequality and c_t is the average of \dot{u}_t with respect to $(\omega^T)^m \wedge \eta$.

Furthermore,

$$\begin{aligned} \int_M |\dot{u}_t - c_t|^2 (\omega^T)^m \wedge \eta &\geq C^{-m} \int_M |\dot{u}_t - c_t|^2 (\omega_{u_t}^T)^m \wedge \eta \\ &\geq C^{-m} \int_M |\dot{u}_t - b_t|^2 (\omega_{u_t}^T)^m \wedge \eta, \end{aligned}$$

where b_t is the average of \dot{u}_t with respect to $(\omega_{u_t}^T)^m \wedge \eta$.

Combining these and integrating gives

$$(69) \quad \begin{aligned} \frac{d}{dt} J_\mu(u_t)|_{t=b} - \frac{d}{dt} J_\mu(u_t)|_{t=a} &\geq \hat{C} \int_a^b \int_M |\dot{u}_t - b_t|^2 (\omega_{u_t}^T)^m \wedge \eta \\ &= \hat{C} d(\omega_{u_a}^T, \omega_{u_b}^T)^2. \end{aligned}$$

Now suppose that u_t is merely a weak $C_w^{1,1}$ geodesics. Then there are smooth ϵ -geodesics u_t^ϵ with $u_t^\epsilon \rightarrow u_t$ as $\epsilon \rightarrow 0$ in the weak- $C_w^{1,1}$ topology. In particular it converges uniformly in $C_b^1([0, 1] \times M)$, so inequality (69) is valid by taking the limit. \square

For the second twisting, let α be a basic, closed, strictly positive $(1,1)$ -form. Define

$$(70) \quad \mathcal{F}^\alpha(u) := \mathcal{E}^\alpha(u) - c_\alpha \mathcal{E}(u),$$

where $c_\alpha > 0$ is chosen so that $\mathcal{F}^\alpha(1) = 0$. From Proposition 3.1.4 \mathcal{F}^α is convex along any weak geodesic in $\text{PSH}(N, \pi^* \omega^T) \cap C^0(N)$. It is further, strictly convex along smooth geodesics. If $\{u_t\}$ is a smooth path in \mathcal{H} a straightforward calculation gives

$$(71) \quad \begin{aligned} \frac{d^2}{dt^2} \mathcal{F}^\alpha(u_t) &= \int_M \left(\ddot{u}_t - \frac{1}{2} |d\dot{u}_t|_{\omega_{u_t}}^2 \right) \left[\text{tr}_{u_t} \alpha - c_\alpha \right] (\omega_{u_t}^T)^m \wedge \eta \\ &\quad + \int_M (d\dot{u}_t \wedge d^c \dot{u}_t, \alpha)_{\omega_{u_t}^T} (\omega_{u_t}^T)^m \wedge \eta. \end{aligned}$$

Proposition 4.2.2. *Suppose (η, ξ, Φ, g) is cscS (respectively Sasaki-extremal), then \mathcal{F}^μ is proper restricted to the orbit \mathcal{O} of H (respectively \mathcal{O}_P of P) and has a unique minimum.*

Proof. Let μ and ν be smooth strictly positive volume forms both with the same total mass on M .

We first consider the case with $\mu = d\mu_{\omega^T} = (\omega^T)^m \wedge \eta$. Let $\phi \in \mathcal{H}$ be normalized so that $\int_M \phi d\mu_{\omega^T} = 0$. We have $\Delta\phi \geq -m$ and by Green's formula

$$(72) \quad \begin{aligned} \phi(x) &= \int_M \Delta\phi(y) G(x, y) d\mu_{\omega^T}(y) + \int_M \phi d\mu_{\omega^T} \\ &\leq -m \int_M G(x, y) d\mu_{\omega^T}(y) = C. \end{aligned}$$

Suppose that μ_{ω^T} and ν have the same total mass on M , with $\nu = f\mu_{\omega^T}$, $f > 0$. Define $\phi_+ = \max(\phi, 0)$ and $\phi_- = \max(-\phi, 0)$, so $\phi = \phi_+ - \phi_-$. We have

$$(73) \quad \int_M \phi_+ d\mu_{\omega^T} = \int_M \phi_- d\mu_{\omega^T}.$$

Then

$$\begin{aligned} |\mathcal{F}^{\mu_{\omega^T}}(\phi) - \mathcal{F}^\nu(\phi)| &= \left| \int_M \phi (d\mu_{\omega^T} - d\nu) \right| \\ &= \left| \int_M \phi (1 - f) d\mu_{\omega^T} \right| \\ &\leq \int_M |\phi f| d\mu_{\omega^T} \\ &= \int_M (\phi_+ + \phi_-) f d\mu_{\omega^T} \leq \hat{C}, \end{aligned}$$

where $\hat{C} > 0$ depends on C , an upper bound on f , and $\text{Vol}(\mu_{\omega^T})$.

Then if μ is any smooth strictly positive measure, with the same total mass as ν , it is easy to see that there is a constant C so that

$$(74) \quad |\mathcal{F}^\mu(\phi) - \mathcal{F}^\nu(\phi)| \leq C, \quad \text{for all } \phi \in \mathcal{H}.$$

If $\mu = \mu_{\omega^T}$ then ω^T is a critical point of \mathcal{F}^μ on the orbit \mathcal{O} (respectively \mathcal{O}_P). Since the exponential map is onto by Proposition 4.2.1 \mathcal{F}^μ is proper on \mathcal{O} (respectively \mathcal{O}_P). By (74) if ν is any smooth strictly positive measure with the same total mass, then \mathcal{F}^ν is proper also. Since for $c > 0$, $\mathcal{F}^{c\nu} = c\mathcal{F}^\nu$, the proof is complete. \square

4.3. Uniqueness of cscS structures. In this section we will prove that the convexity of the K-energy implies the uniqueness of cscS metrics up to automorphisms. More precisely we prove Corollary 2. Since the K-energy \mathcal{M} is not known to be strictly convex on weak geodesics we will modify the K-energy functional.

Let μ be a smooth strictly positive measure invariant under the Reeb flow. We define the modified Mabuchi functional

$$(75) \quad \mathcal{M}^{t\mu}(\phi) := \mathcal{M}(\phi) + t\mathcal{F}^\mu(\phi), \quad \phi \in \mathcal{H}$$

for $t \in [0, \epsilon)$.

The main step in the uniqueness is the following.

Proposition 4.3.1. *Suppose (η, ξ, Φ, g) is cscS. Then there exists a structure $g^*(\eta, \xi, \Phi, g) = (\eta_{\phi_0}, \xi, \Phi_{\phi_0}, g_{\phi_0})$ in the orbit of H and a smooth path, starting at ϕ_0 , $\phi \in C_b^\infty([0, \epsilon) \times M)$ with $\phi_t = \phi(t, \cdot) \in \mathcal{H}$ and ϕ_t a critical point of $\mathcal{M}^{t\mu}$.*

The proof involves a bifurcation technique first used by S. Bando and T. Mabuchi [2] in their uniqueness proof of Kähler-Einstein metrics. More recently, it was used again by X. Chen, M. Păun and Y. Zeng [47] to prove the uniqueness of extremal Kähler metrics.

Proof. By Proposition 4.2.2 there is a unique minimum $\phi_0 \in \mathcal{O}$ of \mathcal{F}^μ restricted to \mathcal{O} .

Define $\mathcal{H}^{k+4, \alpha} = \{\phi \in C_b^{k+4, \alpha} \mid (\omega^T + dd^c \phi)^m \wedge \eta > 0\}$. We define for $k \geq 0$ a map

$$(76) \quad \begin{aligned} \mathcal{G} : \mathcal{H}^{k+4, \alpha}(M) \times [0, \epsilon) &\longrightarrow C_b^{k, \alpha}(M) d\mu \times [0, \epsilon) \\ \mathcal{G}(\phi, t) &= \left((\bar{S} - S_\phi)(\omega_\phi^T)^m \wedge \eta + t(d\mu - (\omega_\phi^T)^m \wedge \eta), t \right) \end{aligned}$$

with differential

$$(77) \quad d\mathcal{G}|_{(\phi_0, 0)}(u, a) = \left(L_{\phi_0} u (\omega_{\phi_0}^T)^m \wedge \eta + a(d\mu - (\omega_{\phi_0}^T)^m \wedge \eta), a \right).$$

But

$$d\mathcal{G}|_{(\phi_0, 0)} : C_b^{k+4, \alpha}(M) \times \mathbb{R} \longrightarrow C_b^{k, \alpha}(M) d\mu \times \mathbb{R}$$

is not surjective or injective if $\mathcal{H} \neq \mathbb{C}$.

Let $d\mu_{\phi_0} = (\omega_{\phi_0}^T)^m \wedge \eta$ and define

$$\mathcal{H}_{\phi_0} := \{u \in C_b^\infty(M) \mid L_{\phi_0}(u) = 0, \int u d\mu_{\phi_0} = 0\}$$

$$\mathcal{H}_{\phi_0, k}^\perp := \{u \in C_b^{k, \alpha}(M) \mid \int uv d\mu_{\phi_0} = 0, \forall v \in \mathcal{H}_{\phi_0}, \int u d\mu_{\phi_0} = 0\}.$$

We have a splitting

$$C_b^{k, \alpha}(M) d\mu = \mathbb{R} d\mu_{\phi_0} \oplus \mathcal{H}_{\phi_0} d\mu_{\phi_0} \oplus \mathcal{H}_{\phi_0, k}^\perp d\mu_{\phi_0}.$$

Since \mathcal{F}^μ has ϕ_0 as a critical point on \mathcal{O} ,

$$d\mu - (\omega_{\phi_0}^T)^m \wedge \eta \in \mathcal{H}_{\phi_0, k}^\perp d\mu_{\phi_0}.$$

Take the first component

$$G(\phi, t) = (\bar{S} - S_\phi)(\omega_\phi^T)^m \wedge \eta + t(d\mu - (\omega_\phi^T)^m \wedge \eta).$$

Define

$$\Pi : \left(\mathbb{R} \oplus \mathcal{H}_{\phi_0} \oplus \mathcal{H}_{\phi_0, k}^\perp \right) \times [0, \epsilon) \rightarrow \left(\mathbb{R} d\mu_{\phi_0} \oplus \mathcal{H}_{\phi_0} d\mu_{\phi_0} \oplus \mathcal{H}_{\phi_0, k}^\perp d\mu_{\phi_0} \right) \times [0, \epsilon),$$

$$(78) \quad \Pi(a + u + w, t) = (ad\mu_{\phi_0} + ud\mu_{\phi_0} + \pi_2 \circ G(\phi_0 + a + u + w, t), t),$$

where π_2 is projection onto $\mathcal{H}_{\phi_0, k}^\perp d\mu_{\phi_0}$. Since $d\Pi|_{(0,0)}$ is bijective, we apply the implicit function theorem to get $\epsilon > 0$ so that $\|u\|_{C_b^{k, \alpha}} < \epsilon$, $t < \epsilon$ implies that there is $\Psi(u, t) \in \mathcal{H}_{\phi_0, k}^\perp$ such that

$$(79) \quad \pi_2 \circ G(\phi_0 + u + \Psi(u, t), t) = 0.$$

Differentiating (79) with respect to t gives

$$(80) \quad L_{\phi_0} \frac{\partial \Psi}{\partial t}|_{(0,0)} (\omega_{\phi_0}^T)^m \wedge \eta + (d\mu - (\omega_{\phi_0}^T)^m \wedge \eta) = 0,$$

while differentiating with respect to u gives

$$(81) \quad \frac{\partial \Psi}{\partial u}|_{(0,0)}(v) = 0, \quad \text{for all } v \in \mathcal{H}_{\phi_0}.$$

Define

$$(82) \quad \begin{aligned} P(u, t) &:= \pi_1 \circ G(\phi_0 + u + \Psi(u, t), t) \\ \tilde{P}(u, t) &:= \frac{P(u, t)}{t}, \quad t \in (0, \epsilon), \quad \tilde{P}(u, 0) := \lim_{t \rightarrow 0^+} \frac{P(u, t)}{t} = \frac{\partial P}{\partial t}|_{(u,0)}. \end{aligned}$$

To complete the proof we will need a technical lemma. For $\phi \in \mathcal{H}$ we define a bilinear form $B_\phi(\cdot, \cdot)$ on $C_b^\infty(M)$

$$\begin{aligned} B_\phi(u, v) &:= (\partial \bar{\partial} v, \partial \bar{\partial} \Delta_\phi u)_\phi + \Delta_\phi (\partial \bar{\partial} v, \partial \bar{\partial} u)_\phi + (\partial \bar{\partial} \Delta_\phi v, \partial \bar{\partial} u)_\phi \\ &\quad + g^{T\bar{p}\bar{q}} u_{,\bar{\alpha}p} v_{,\beta\bar{q}} (\text{Ric}_\phi^T)^{\bar{\alpha}\beta} + g^{T\bar{p}q} u_{,\bar{p}\alpha} v_{,q\bar{\beta}} (\text{Ric}_\phi^T)^{\alpha\bar{\beta}} \end{aligned}$$

We refer the reader to [47] for the proof of the following.

Lemma 4.3.2. *Let $(\eta_\phi, \xi, \Phi_\phi, g_\phi)$ be Sasaki-extremal, and $v \in \mathcal{H}_\phi$, then*

$$L_\phi(\partial v, \bar{\partial} u)_\phi = (\partial v, \bar{\partial} L_\phi u)_\phi + B_\phi(v, u),$$

for all $u \in C_b^\infty(M)$.

We compute

$$\begin{aligned} \tilde{P}(u, 0) &= \frac{\partial}{\partial t} P|_{(u,0)} \\ &= \pi_1 \left(L_{\phi_0+u+\Psi(u,0)} \frac{\partial \Psi}{\partial t}|_{(u,0)} (\omega_{\phi_0+u+\Psi(u,0)}^T)^m \wedge \eta \right. \\ &\quad \left. - (S_{\phi_0+u+\Psi(u,0)} - \bar{S}) \Delta_{\phi_0+u+\Psi(u,0)} \frac{\partial \Psi}{\partial t}|_{(u,0)} (\omega_{\phi_0+u+\Psi(u,0)}^T)^m \wedge \eta \right. \\ &\quad \left. + (d\mu - (\omega_{\phi_0+u+\Psi(u,0)}^T)^m \wedge \eta) \right). \end{aligned}$$

Let $w = \frac{\partial \Psi}{\partial t}|_{(0,0)}$. We compute the differential in the \mathcal{H}_{ϕ_0} direction. For $v \in \mathcal{H}_{\phi_0}$

$$\begin{aligned} \frac{\partial}{\partial u} \tilde{P}|_{(0,0)}(v) &= \pi_1 \left(\frac{\partial}{\partial u} \left\{ L_{\phi_0+u+\Psi(u,0)} \frac{\partial \Psi}{\partial t}|_{(u,0)} (\omega_{\phi_0+u+\Psi(u,0)}^T)^m \wedge \eta \right\} (v) - \Delta_{\phi_0} v (\omega_{\phi_0}^T)^m \wedge \eta \right) \\ &= \pi_1 \left(-B_{\phi_0}(v, w) (\omega_{\phi_0}^T)^m \wedge \eta + L_{\phi_0}(w) \Delta_{\phi_0} v (\omega_{\phi_0}^T)^m \wedge \eta - \Delta_{\phi_0} v (\omega_{\phi_0}^T)^m \wedge \eta \right) \end{aligned}$$

By Lemma 4.3.2

$$(83) \quad \begin{aligned} \frac{\partial}{\partial u} \tilde{P}|_{(0,0)}(v) &= \pi_1 \left(-L_{\phi_0}(\partial v, \bar{\partial} w)_{\phi_0} (\omega_{\phi_0}^T)^m \wedge \eta + (\partial v, \bar{\partial} L_{\phi_0} w) \right. \\ &\quad \left. + L_{\phi_0}(w) \Delta_{\phi_0} v (\omega_{\phi_0}^T)^m \wedge \eta - \Delta_{\phi_0} v (\omega_{\phi_0}^T)^m \wedge \eta \right). \end{aligned}$$

We define $f \in C_b^\infty(M)$, $f > 0$, by

$$(84) \quad f = \frac{d\mu}{(\omega_{\phi_0}^T)^m \wedge \eta}.$$

Then (80) gives

$$L_{\phi_0} w = \frac{-d\mu}{(\omega_{\phi_0}^T)^m \wedge \eta} + 1 = -f + 1.$$

Substituting into (83) we have

$$(85) \quad \frac{\partial}{\partial u} \tilde{P}|_{(0,0)}(v) = -\pi_1 \left((\partial v, \bar{\partial} f)_{\phi_0} (\omega_{\phi_0}^T)^m \wedge \eta + f(\Delta_{\phi_0} v)(\omega_{\phi_0}^T)^m \wedge \eta \right).$$

And

$$\begin{aligned} \left\langle \frac{\partial}{\partial u} \tilde{P}|_{(0,0)}(v), v \right\rangle_{L^2} &= - \int_M v(\partial v, \bar{\partial} f)_{\phi_0} (\omega_{\phi_0}^T)^m \wedge \eta - \int_M v f(\Delta_{\phi_0} v)(\omega_{\phi_0}^T)^m \wedge \eta \\ &= \int_M (\partial v, \bar{\partial} v)_{\phi_0} f(\omega_{\phi_0}^T)^m \wedge \eta = \int_M (\partial v, \bar{\partial} v)_{\phi_0} d\mu > 0 \end{aligned}$$

unless $v = 0$. Thus $\frac{\partial}{\partial u} \tilde{P}|_{(0,0)} : \mathcal{H}_{\phi_0} \rightarrow \mathcal{H}_{\phi_0} d\mu_{\phi_0}$ is an isomorphism. By the implicit function theorem there exists $u_t \in \mathcal{H}_{\phi_0}$, $t \in [0, \epsilon)$ so that $\phi_0 + u_t + \Psi(u_t, t)$ is the required solution. \square

We now prove Corollary 2. Let $(\eta_0, \xi, \Phi_0, g_0)$ and $(\eta_1, \xi, \Phi_1, g_1)$ be two cscS structures. By Proposition 4.3.1 there exists a smooth path $\{\phi_s^0 \mid s \in [0, \epsilon)\} \in \mathcal{H}_{\omega^T}$ such that $\omega_{\phi_0^0}^T$ is in the orbit \mathcal{O}^0 of $(\eta_0, \xi, \Phi_0, g_0)$. Similarly, there exists a smooth path $\{\phi_s^1 \mid s \in [0, \epsilon)\} \in \mathcal{H}_{\omega^T}$ such that $\omega_{\phi_1^0}^T$ is in the orbit \mathcal{O}^1 of $(\eta_1, \xi, \Phi_1, g_1)$. For each $s \in [0, \epsilon)$ let $\{u_t^s \mid 0 \leq t \leq 1\}$ be the weak $C_w^{1,1}$ geodesic with $u_0^s = \phi_s^0$ and $u_1^s = \phi_s^1$.

By Lemma 3.3.4 we have $\frac{d}{dt} \mathcal{M}^{s\mu}(u_t^s)|_{t=0^+} \geq 0$, and similarly $\frac{d}{dt} \mathcal{M}^{s\mu}(u_t^s)|_{t=1^-} \leq 0$. Thus

$$\frac{d}{dt} \mathcal{M}^{s\mu}(u_t^s)|_{t=1^-} - \frac{d}{dt} \mathcal{M}^{s\mu}(u_t^s)|_{t=0^+} \leq 0.$$

But by strict convexity of $\mathcal{M}^{s\mu}(u_t^s)$, this must be > 0 , unless $\omega_{u_0^s}^T = \omega_{u_1^s}^T$ for all $s \in (0, \epsilon)$. Thus $\omega_{\phi_0^0}^T = \omega_{\phi_1^0}^T$.

4.4. Uniqueness of Sasaki-extremal structures. Sasaki-extremal structures are characterized as critical points of a modified Mabuchi functional \mathcal{M}^V . Suppose that (η, ξ, Φ, g) is a Sasakian structure with $G = \text{Aut}(g, \eta, \xi, \Phi)_0 \subset \text{Fol}(M, \mathcal{F}_\xi, J)$ a maximal compact subgroup. We will define this functional and prove uniqueness from convexity using a deformation technique as in the cscS case. Let V be the extremal vector field, defined in (66), and let h_ϕ^V be its holomorphy potential with respect to ω_ϕ^T , $\phi \in \mathcal{H}^G$, normalized by $\int_M h_\phi^V d\mu_\phi = 0$.

Proposition 4.4.1. *Suppose $W \in \mathfrak{hol}^T(\xi, J)_0$ has normalized holomorphy potential h^W with respect to ω^T . Then the normalized holomorphy potential of W with respect to ω_ϕ^T is*

$$h_\phi^W = h^W + W(\phi).$$

The proof is straight forward. See for example [27]. Note that for the extremal vector field V h_ϕ^V is real valued for $\phi \in \mathcal{H}^G$, since $\text{Im } W \in \mathfrak{g}$.

We define \mathcal{E}^V on \mathcal{H}^G as the unique functional with

$$d\mathcal{E}^V|_\phi(\dot{\phi}) = \int_M \dot{\phi} h_\phi^V(\omega_\phi^T)^m \wedge \eta.$$

This form is well-known to be closed. Thus the definition

$$(86) \quad \mathcal{E}^V(\phi) = \int_0^1 \int_M \dot{\phi}_t h_{\phi_t}^V(\omega_{\phi_t}^T)^m \wedge \eta,$$

where $\{\phi_t \mid 0 \leq t \leq 1\}$ is a smooth path in \mathcal{H}^G with $\phi_0 = 0$ and $\phi_1 = \phi$, is path independent. There is a closed form formula for \mathcal{E}^V , found by integrating (86) along linear paths,

$$(87) \quad \mathcal{E}^V(\phi) = \frac{1}{(m+1)(m+2)} \int_M \phi \sum_{k=0}^m ((n-k+1)h^V + (k+1)h_\phi^V)(\omega^T)^k \wedge (\omega_\phi)^{m-k} \wedge \eta.$$

One can then uniquely extend \mathcal{E}^V to $\text{PSH}(M, \omega^T) \cap C^1(M)$ by (87). This functional is then continuous in C^1 by Theorem 2.3.1.

Proposition 4.4.2. *Let $\{u_t \mid 0 \leq t \leq 1\}$ be a $C_w^{1,1}$ weak geodesic between $u_0, u_1 \in \mathcal{H}$. Then \mathcal{E}^V is linear along $\{u_t\}$, that is $\frac{d^2}{dt^2} \mathcal{E}(u_t) = 0$.*

Proof. Let $\{u_t^\epsilon \mid 0 \leq t \leq 1\}$ for $\epsilon > 0$ be ϵ -geodesics. Thus they are smooth and increase monotonically to $\{u_t\}$ as $\epsilon \searrow 0$. The paths $\{u_t^\epsilon \mid 0 \leq t \leq 1\}$ converge weakly in $C_w^{1,1}$ to $\{u_t \mid 0 \leq t \leq 1\}$; in particular, they converge in C^1 .

Given any smooth path $\{w_t\}$ in \mathcal{H} we have

$$(88) \quad \frac{d}{dt} \int_M \dot{w}_t h_{w_t}^V(\omega_{w_t}^T)^m \wedge \eta = \int_M (\ddot{w}_t - \frac{1}{2} |d\dot{w}_t|_{\omega_{w_t}^T}^2) h_{w_t}^V(\omega_{w_t}^T)^m \wedge \eta.$$

Thus

$$(89) \quad \frac{d}{dt} \int_M \dot{u}_t^\epsilon h_{u_t^\epsilon}^V(\omega_{u_t^\epsilon}^T)^m \wedge \eta = \epsilon \int_M h_{u_t^\epsilon}^V(\omega_{u_t^\epsilon}^T)^m \wedge \eta.$$

By Theorem 2.3.1

$$(90) \quad \frac{d}{dt} \mathcal{E}^V(u_t^\epsilon) = \int_M \dot{u}_t^\epsilon h_{u_t^\epsilon}^V(\omega_{u_t^\epsilon}^T)^m \wedge \eta \longrightarrow \int_M \dot{u}_t h_{u_t}^V(\omega_{u_t}^T)^m \wedge \eta.$$

And from (89) there is a constant $C > 0$ so that

$$-\epsilon C \leq \frac{d}{dt} \mathcal{E}^V(u_t^\epsilon)|_{t=b} - \frac{d}{dt} \mathcal{E}^V(u_t^\epsilon)|_{t=a} \leq \epsilon C.$$

Thus $\frac{d}{dt} \mathcal{E}^V(u_t)$ is constant. \square

We define

$$(91) \quad \mathcal{M}^V(\phi) := \mathcal{M}(\phi) + \mathcal{E}^V(\phi), \quad \phi \in \mathcal{H}^G.$$

Since we do not have strict convexity for \mathcal{M}^V it will again be necessary to modify it. So we define

$$(92) \quad \mathcal{M}^{V,t\mu}(\phi) := \mathcal{M}^V(\phi) + t\mathcal{F}^\mu, \quad \phi \in \mathcal{H}^G,$$

for $t \in [0, \epsilon]$, where we now assume that μ is a smooth strictly positive G -invariant volume form.

Uniqueness of Sasaki-extremal structures will follow from the following, whose proof is similar to Proposition 4.3.1

Proposition 4.4.3. *Suppose (η, ξ, Φ, g) is Sasaki-extremal. Then there exists a structure $g^*(\eta, \xi, \Phi, g) = (\eta_{\phi_0}, \xi, \Phi_{\phi_0}, g_{\phi_0})$ in the orbit of P and a smooth path, starting at ϕ_0 , $\phi \in C_b^\infty([0, \epsilon) \times M)$ with $\phi_t = \phi(t, \cdot) \in \mathcal{H}$ and ϕ_t a critical point of $\mathcal{M}^{V, t\mu}$.*

Proof. As before, by Proposition 4.2.2 there is a unique minimum $\phi_0 \in \mathcal{H}^G$ of \mathcal{F}^μ restricted to the orbit \mathcal{O}_P of P .

If we denote by h^V the normalized holomorphy potential for V with respect to ω^T , then $h^V = S_g - \bar{S}$, while for $\phi \in \mathcal{H}^G$ the holomorphy potential with respect to ω_ϕ^T is

$$h_\phi^V = S_g - \bar{S} + V(\phi).$$

We define $\mathcal{H}_{k+4, \alpha}^G = \{\phi \in C_b^{k+4, \alpha}(M)^G \mid (\omega^T + dd^c \phi)^m \wedge \eta > 0\}$. And we define a map

$$(93) \quad \begin{aligned} \mathcal{G}^V : \mathcal{H}_{k+4, \alpha}^G \times [0, \epsilon) &\longrightarrow C_b^{k, \alpha}(M)^G d\mu \times [0, \epsilon) \\ \mathcal{G}^V(\phi, t) &= \left((\bar{S} - S_\phi + h_\phi^V)(\omega_\phi^T)^m \wedge \eta + t(d\mu - (\omega_\phi^T)^m \wedge \eta), t \right) \end{aligned}$$

with differential

$$(94) \quad d\mathcal{G}^V|_{(\phi_0, 0)}(u, a) = \left(L_{\phi_0} u (\omega_{\phi_0}^T)^m \wedge \eta + a(d\mu - (\omega_{\phi_0}^T)^m \wedge \eta), a \right).$$

But as before But

$$d\mathcal{G}^V|_{(\phi_0, 0)} : C_b^{k+4, \alpha}(M)^G \times \mathbb{R} \longrightarrow C_b^{k, \alpha}(M)^G d\mu \times \mathbb{R}$$

is in general not surjective or injective. Let $d\mu_{\phi_0} = (\omega_{\phi_0}^T)^m \wedge \eta$ and define

$$\begin{aligned} \mathcal{H}_{\phi_0}^G &:= \{u \in C_b^\infty(M)^G \mid L_{\phi_0}(u) = 0, \int u d\mu_{\phi_0} = 0\} \\ \mathcal{H}_{\phi_0, k}^{\perp, G} &:= \{u \in C_b^{k, \alpha}(M) \mid \int uv d\mu_{\phi_0} = 0, \forall v \in \mathcal{H}_{\phi_0}, \int u d\mu_{\phi_0} = 0\}. \end{aligned}$$

We have a splitting

$$C_b^{k, \alpha}(M)^G d\mu = \mathbb{R} d\mu_{\phi_0} \oplus \mathcal{H}_{\phi_0}^G d\mu_{\phi_0} \oplus \mathcal{H}_{\phi_0, k}^{\perp, G} d\mu_{\phi_0}.$$

Define

$$G^V(\phi, t) = (\bar{S} - S_\phi + h_\phi^V)(\omega_\phi^T)^m \wedge \eta + t(d\mu - (\omega_\phi^T)^m \wedge \eta).$$

As before the implicit function theorem gives $\Psi(u, t) \in \mathcal{H}_{\phi_0, k}^{\perp, G}$ such that

$$\pi_2 \circ G^V(\phi_0 + u + \Psi(u, t), t) = 0.$$

Differentiating with respect to t gives

$$(95) \quad L_{\phi_0} \frac{\partial \Psi}{\partial t}|_{(0, 0)} (\omega_{\phi_0}^T)^m \wedge \eta + (d\mu - (\omega_{\phi_0}^T)^m \wedge \eta) = 0,$$

while differentiating with respect to u gives

$$(96) \quad \frac{\partial \Psi}{\partial u}|_{(0, 0)}(v) = 0, \quad \text{for all } v \in \mathcal{H}_{\phi_0}^G.$$

Define

$$(97) \quad \begin{aligned} P(u, t) &:= \pi_1 \circ G^V(\phi_0 + u + \Psi(u, t), t) \\ \tilde{P}(u, t) &:= \frac{P(u, t)}{t}, \quad t \in (0, \epsilon), \quad \tilde{P}(u, 0) := \lim_{t \rightarrow 0^+} \frac{P(u, t)}{t} = \frac{\partial P}{\partial t}|_{(u, 0)}. \end{aligned}$$

Set $w = \frac{\partial \Psi}{\partial t}|_{(0, 0)}$. Then the same computation as in Proposition 4.3.1

$$(98) \quad \begin{aligned} \frac{\partial}{\partial u} \tilde{P}|_{(0, 0)}(v) &= \pi_1 \left((\partial v, \bar{\partial} L_{\phi_0} w) \right. \\ &\quad \left. + L_{\phi_0}(w) \Delta_{\phi_0} v (\omega_{\phi_0}^T)^m \wedge \eta - \Delta_{\phi_0} v (\omega_{\phi_0}^T)^m \wedge \eta \right). \end{aligned}$$

Using that (95) gives $L_{\phi_0} w = -f + 1 \in C_b^\infty(M)^G$, where f is defined in (84), we get as before

$$\left\langle \frac{\partial}{\partial u} \tilde{P}|_{(0, 0)}(v), v \right\rangle_{L^2} = \int_M (\partial v, \bar{\partial} v)_{\phi_0} d\mu > 0$$

unless $v = 0$ for $v \in \mathcal{H}_{\phi_0}^G$.

By the implicit function theorem there exists $u_t \in \mathcal{H}_{\phi_0}^G$ with $P(u_t, t) = 0$ and $G^V(\phi_0 + u_t + \Phi(u_t, t), t) = 0$. \square

We now prove Corollary 4. First we apply the last part of Theorem 4.1.1 act by an element of $\text{Fol}(M, \mathcal{F}_\xi, J)$ so that $\text{Aut}(\eta_0, \xi, \Phi_0, g_0)_0 = \text{Aut}(\eta_1, \xi, \Phi_1, g_1)_0 = G$. The rest of the proof of Corollary 4 is nearly identical to that of Corollary 2. One just makes use that \mathcal{E}^V is affine along weak geodesics.

We now prove Corollary 5. Let $\{\phi_t\}$ be the weak $C_w^{1,1}$ geodesic connecting $\phi_0, \phi_1 \in \mathcal{H}^G$. Lemma 3.3.4 gives

$$(99) \quad \frac{d}{dt} \mathcal{M}(\phi_t)|_{t=0^+} \geq - \int_M (S_{\phi_0}^T - \bar{S}^T - h_{\phi_0}^V) \frac{d\phi_t}{dt}|_{t=0^+} (\omega_{\phi_0}^T)^m \wedge \eta.$$

Note that for $\phi \in \mathcal{H}^G$, if we denote by S_ϕ^G the reduced scalar curvature of $(\eta_\phi, \xi, \Phi_\phi, g_\phi)$, then

$$S_\phi^G = S_\phi^T - \bar{S}^T - h_\phi^V.$$

The rest of the proof follows from (99) just as in the proof of Corollary 3.

Remark 4.4.4. Corollary 5 can be easily generalized to any compact group $K \subset G$ such that $\text{Im } V \in K$. The inequality then applies to two potentials $\phi_0, \phi_1 \in \mathcal{H}^K$ and in the righthand side the Calabi functional $\text{Cal}_{M, \xi}^K(\phi) = \int_M (S_\phi^K)^2 d\mu_\phi$, where the reduced scalar curvature S_ϕ^K is defined as in (67).

We now prove Corollary 6. Let $(\eta_0, \xi_0, \Phi_0, g_0), (\eta_1, \xi_1, \Phi_1, g_1)$ be two Sasaki-extremal structures with Reeb foliation \mathcal{F} with its given transversely holomorphic structure.

We consider the *leafwise cohomology* defined by the complex

$$0 \rightarrow C^\infty(M) \xrightarrow{d^\mathcal{F}} C^\infty(\Lambda^1 \mathcal{F}) \rightarrow 0,$$

where for $f \in C^\infty(M)$, $d^\mathcal{F} f = df|_{T\mathcal{F}}$. Any contact form η of a Sasakian structure compatible with \mathcal{F} , with its holomorphic structure defines a class $[\eta]_\mathcal{F} \in H^1(\mathcal{F})$. $H^1(\mathcal{F})$ can be identified with the component $E_1^{0,1}(\mathcal{F})$ of the E_1 -term of the spectral sequence associated with \mathcal{F} (cf. [32]). Then $E_2^{0,1}(\mathcal{F}) \subseteq E_1^{0,1}(\mathcal{F})$ consists of d_1 -closed elements. By [32, Cor. 4.7] $\dim E_2^{0,1}(\mathcal{F}) = 1$. Since $d_1[\eta]_\mathcal{F} = 0$ for any

contact form of a compatible Sasakian structure, $E_2^{0,1}(\mathcal{F})$ is generated by $[\eta]_{\mathcal{F}}$. Thus $[\eta_0]_{\mathcal{F}} = a[\eta_1]_{\mathcal{F}}$ for $a \neq 0$, and there exists $f \in C^\infty(M)$ with

$$\eta_0 - a\eta_1 - df|_{T\mathcal{F}} = 0.$$

Define $\eta_t = \eta_0 + t(a\eta_1 - \eta_0)$ and define $X_t \in C^\infty(T\mathcal{F})$ by $\eta_t(X_t) = f$. Then if Ψ_t is the flow of X_t , an easy computation shows

$$\frac{d}{dt}\Psi_t^*\eta_t|_{T\mathcal{F}} = 0.$$

So $\eta_0|_{T\mathcal{F}} = a\Psi_1^*\eta_1|_{T\mathcal{F}}$. It follows that $a > 0$ and $\Psi_1^*(\eta_1, \xi_1, \Phi_1, g_1) = (a^{-1}\hat{\eta}, a\xi_0, \hat{\Phi}, \hat{g})$.

Given a Sasakian structure (η, ξ, Φ, g) the *transverse homothety* by $a > 0$ is the Sasakian structure $(\eta_a, \xi_a, \Phi, g_a)$ with

$$\eta_a = a\eta, \quad \xi_a = a^{-1}\xi, \quad \Psi_a = \Psi, \quad g_a = ag + (a^2 - a)\eta \otimes \eta.$$

Since $g_a^T = ag^T$ the transverse Ricci curvature is unchanged $\text{Ric}_{g_a^T} = \text{Ric}_{g^T}$. So a transverse homothety of a cscS (respectively Sasaki-extremal) structure is cscS (respectively Sasaki-extremal).

Performing a transverse homothety by $a > 0$ we get $(\hat{\eta}, \xi_0, \hat{\Phi}, \hat{g}_a)$. By Lemma 2.2.3

$$\hat{\eta} = \eta_0 + 2d^c\phi + d\psi + \alpha,$$

where $\alpha \in \mathcal{H}_{g^T}^1$ is a transversal harmonic 1-form. The exact component $d\psi$ is just given by a gauge transformation. More precisely, if $b = \exp(-\psi\xi_0)$, then $b^*\hat{\eta} = \eta_0 + 2d^c\phi + \alpha$. By Corollary 4 there is a $g \in \text{Fol}(\mathcal{F}, J)$ with $g^*b^*\frac{1}{2}d\hat{\eta} = \frac{1}{2}d\eta_0$.

4.5. Results on the α -twisted case. We prove uniqueness results for twisted constant scalar curvature metrics and, more generally, twisted extremal metrics. These metrics have been of interest in Kähler geometry [44, 18] as a possible approach to the general existence problem of constant scalar curvature metrics and their connection to geometric stability.

In this section α will be any smooth, basic, strictly positive $(1, 1)$ -form on M . A Sasakian structure (η, ξ, Φ, g) has constant α -twisted scalar curvature if

$$S_g^T - \text{tr}_{\omega^T} \alpha = C_\alpha,$$

where C_α is a constant that depends only on the Sasakian structure and the basic cohomology class $[\alpha] \in H_b^2(M, \mathbb{R})$. These metrics are precisely the Sasakian structures in $\mathcal{S}(\xi, J)$ which are critical points of

$$(100) \quad \mathcal{M}^\alpha(\phi) = \mathcal{M}(\phi) + \mathcal{F}^\alpha(\phi), \quad \phi \in \mathcal{H}.$$

Theorem 4.5.1. *Any two α -twisted constant scalar curvature structures in $\mathcal{S}(\xi, J)$ have the same transversal Kähler metric.*

Proof. As before, we consider a perturbed Mabuchi functional

$$(101) \quad \mathcal{M}^{\alpha, t\mu} := \mathcal{M}^\alpha + t\mathcal{F}^\mu.$$

Suppose that $\phi_0 \in \mathcal{H}$ is such that $\int \phi_0 d\mu_{\phi_0} = 0$ and $\omega_{\phi_0}^T$ has constant α -twisted scalar curvature. Define

$$G : \tilde{\mathcal{H}}^{k+4, \alpha} \rightarrow \tilde{C}_b^{k, \alpha}(M) d\mu_{\phi_0},$$

where $\tilde{\mathcal{H}}^{k+4, \alpha} = \{\phi \in C_b^{k+4, \alpha} \mid (\omega^T + dd^c\phi)^m \wedge \eta > 0, \int \phi d\mu_{\phi_0} = 0\}$, and $\tilde{C}_b^{k, \alpha}(M) d\mu_{\phi_0}$ is the subspace with integral zero, by

$$G(\phi) = (\bar{S}^T - S_\phi^T)(\omega_\phi^T)^m \wedge \eta + m\alpha \wedge (\omega_\phi^T)^{m-1} \eta - C_\alpha(\omega_\phi^T)^m \wedge \eta.$$

We compute

$$(102) \quad dG|_{\phi_0}(u) = \left(L_{\phi_0}(u) + \sqrt{-1}(\bar{\partial}u, \bar{\partial}^* \alpha)_{\phi_0} - (\sqrt{-1}\partial\bar{\partial}u, \alpha)_{\phi_0} \right) (\omega_{\phi_0}^T)^m \wedge \eta,$$

where we have used that $S_{\phi_0}^T$ satisfies the α -twisted cscS equation. Integrating by parts gives

$$\left\langle dG|_{\phi_0}(u), u \right\rangle_{L^2} = \int_M u L_{\phi_0}(u) d\mu_{\phi_0} + \int_M (\sqrt{-1}\partial u \wedge \bar{\partial}, \alpha)_{\phi_0} d\mu_{\phi_0} > 0,$$

unless u is constant. Since $dG|_{\phi_0}$ transversely is elliptic, it is Fredholm. And since it differs from $\Delta_{\phi_0}^2$ by a compact operator its index is zero, and is therefore an isomorphism.

Define

$$\begin{aligned} \mathcal{G} : \tilde{\mathcal{H}}^{k+4, \alpha} \times [0, \epsilon) &\rightarrow \tilde{C}_b^{k, \alpha}(M) d\mu_{\phi_0} \times [0, \epsilon), \\ \mathcal{G}(\phi, t) &= (G(\phi) + t(d\mu - (\omega_{\phi}^T)^m \wedge \eta), t). \end{aligned}$$

Thus

$$d\mathcal{G}|_{\phi_0}(u, a) = (dG|_{\phi_0}(u) + a(d\mu - (\omega_{\phi}^T)^m \wedge \eta), a)$$

is an isomorphism. The implicit function theorem then gives a path $\{\phi_t \mid t \in [0, \epsilon)\}$ in \mathcal{H} with ϕ_t a critical point of $\mathcal{M}^{\alpha, t\mu}$. The proof is completed just as that of Corollary 2 using that \mathcal{F}^{α} is convex along weak geodesics. \square

A Sasakian structure (η, ξ, Φ, g) is α -twisted extremal if

$$S_g^T - \text{tr}_{\omega^T} \alpha = \mathcal{H}_g.$$

Thus the left hand side is $h^V + C_{\alpha}$, where h^V is the normalized holomorphy potential. Since it is a real potential, $\partial^{\#} h^V = V \in \mathfrak{h}^{\text{Ham}}$ has $\text{Im } V$ preserving (η, ξ, Φ, g) .

Lemma 4.5.2. *We have $\mathcal{L}_{\text{Im } V} \alpha = 0$. Thus if $K \subset \text{Aut}(\eta, \xi, \Phi, g)$ is the closure of $\{\exp(s\xi), \exp(t \text{Im } V) \mid s, t \in \mathbb{R}\}$. Then α is K -invariant.*

Proof. Averaging α with respect to K gives a K -invariant $\hat{\alpha}$ with $[\hat{\alpha}] = [\alpha]$. So there exists $\psi \in C_b^{\infty}(M)$ with $\alpha = \hat{\alpha} + dd^c \psi$. Then the α -twisted extremal equation becomes

$$S_g - \text{tr}_{\omega^T} \hat{\alpha} - \Delta_{\omega^T} \psi = h^V + C_{\alpha}.$$

Taking the Lie derivative gives $\Delta_{\omega^T} \text{Im } V(\psi) = 0$, which implies $\text{Im } V(\psi) = 0$. \square

We are able to prove a partial uniqueness result for α -twisted extremal structures by modifying the proof of Theorem 4.5.1.

Theorem 4.5.3. *Any two α -twisted extremal structures in $\mathcal{S}(\xi, J)$ with $\partial^{\#}(S_g^T - \text{tr}_{\omega^T} \alpha) = V$ have the same transversal Kähler metric.*

The proof goes through just as Theorem 4.5.1, mutatis mutandis. Unlike the untwisted extremal case there is no reason for the vector field V to be an invariant of the polarization.

We remark that versions of Propositions 4.3.1 (respectively 4.4.3) involving the deformed Mabuchi functional $\mathcal{M}^{t\alpha} = \mathcal{M} + t\mathcal{F}^{\alpha}$ (respectively $\mathcal{M}^{V, t\alpha} = \mathcal{M}^V + \mathcal{F}^{\alpha}$) can be proved following the same method as the above proofs, as long as $[\alpha] = [\omega^T]$ in basic cohomology.

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